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Soliton complexes and flat-top nonlinear modes in optical lattices

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ABSTRACT We describe a continuous analog of the quasirectangular flat-top nonlinear modes earlier found for discrete nonlinear models. We show that these novel nonlinear modes can be understood as multi-soliton complexes with either in-phase or out-of-phase neighboring solitons trapped by the periodic potential of the lattice. We demonstrate a link between the flat-top states and the truncated nonlinear Bloch waves, and discuss how these nonlinear localized modes can be monitored experimentally in photonics and Bose–Einstein condensates.

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1 Introduction

Nonlinear periodic photonic structures such as arrays of optical waveguides recently attracted a lot of interest due to the unique ways they offer for controlling light [1]. Periodic modulation of the refractive index breaks the translational invariance and produces effective discreteness in a continuous system, leading to nontrivial modification of wave diffraction and affecting nonlinear propagation of light. This opens up many novel possibilities for manipulating light propagation, including light localization in the form of discrete optical solitons [2].

Discrete optical solitons were introduced as spatially localized nonlinear modes of weakly coupled optical waveguides [3], which exist due to a balance between discrete diffraction in the array and the nonlinear response of the waveguide material. A standard theoretical approach to study discrete optical solitons is based on the tight-binding approximation and the effective discrete nonlinear model – the discrete nonlinear Schrödinger (NLS) equation employed for analyzing stationary localized solutions [4]. This approach allows the prediction of many types of nonlinear modes in discrete systems and then the verification of their existence in the continuous nonlinear models [5].

Discrete spatial solitons are localized due to total internal reflection allowing the trapping of an optical beam by a few neighboring waveguides suppressing diffraction-induced light spreading, as was first demonstrated experimentally by Eisenberg et al. [6]. Since then, discrete solitons have been demonstrated in several other physical systems, including dynamic holographic gratings induced optically in photorefractive crystals [7, 8] and liquid-crystal waveguides with pattern electrodes [9].

In spite of experimental observations and elaborated theory, the flat-top nonlinear states introduced earlier as specific quasirectangular solutions of the discrete NLS equation [10] have neither been discussed for the continuous models of the periodic photonic systems nor demonstrated experimentally in photonics. Thus, the important questions remain: can the flat-top modes exist in the continuous models and how do they link to other types of nonlinear modes of nonlinear periodic systems?

In this paper, we describe a continuous analog of the nonlinear flat-top modes earlier found for the discrete NLS equation [4, 10]. We demonstrate that these nonlinear modes can be understood as complexes of many almost identical solitons placed in phase or out of phase at the minima of the optical lattice [11], and thus these modes provide an important missing link between the concepts of lattice solitons [5] and nonlinear Bloch waves [12] discussed earlier for the continuous periodic nonlinear systems. Finally, we mention that the recent observation of nonlinear self-trapping of matter waves in one-dimensional optical lattices [13] can be interpreted as the excitation of the broad nonlinear states in the gaps of the matter-wave band-gap spectrum.

2 Nonlinear self-trapping in periodic structures

We consider propagation of optical beams along the z axis in a cubic nonlinear medium with modulation of the linear refractive index along the transverse x axis, described by the continuous NLS equation for the dimensionless complex field envelope amplitude ψ,

$$\mathrm{i}\frac{\partial\psi}{\partial z}+\frac{\partial^2\psi}{\partial x^2}-V(x)\psi+\sigma|\psi|^2\psi=0\,, \tag{1}$$

where the longitudinal z and transverse x coordinates are scaled to the diffraction length and the input beam width, respectively. The external periodic potential $V(x)$ in Eq. (1) describes the transverse profile of the refractive index in a periodic photonic lattice, taken as

$$V(x)=V_0\sin^2(\pi x/a)\,, \tag{2}$$

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where a is the lattice spacing (below, we take $a = \pi$), and the amplitude V_0 is proportional to the depth of the refractive-index modulation. The parameter $\sigma = \pm 1$ defines the nonlinearity sign (focusing/defocusing). The similar model at $\sigma = -1$ describes the dynamics of the elongated cigar-shaped Bose–Einstein condensates in a one-dimensional optical lattice [12].

We look for stationary nonlinear modes in the form

$$\psi(x, z) = \Psi(x) \exp(\mathrm{i}\beta z)\,, \tag{3}$$

where the real envelope $\Psi(x)$ satisfies the equation

$$\frac{\mathrm{d}^2\Psi}{\mathrm{d}x^2} - V_0 \sin^2 x\, \Psi + \sigma|\Psi|^2\Psi = \beta\Psi\,. \tag{4}$$

For $\sigma = 0$ this reduces to a linear eigenvalue problem with stationary states given by linear Bloch waves $\Psi(x) = \phi_k(x)\exp(\mathrm{i}kx)$, where the wavevector k determines the location in the Brillouin zone and $\phi_k(x) = \phi_k(x+\pi)$ is a periodic function with the periodicity of the lattice. The linear Bloch wave spectrum is shown in Fig. 1 with the associated linear band gaps.

We solve Eq. (4) numerically with a special target, to search for various types of broad nonlinear localized modes that resemble the so-called quasirectangular flat-top states of the discrete NLS model [4, 10]. Figure 1 shows some examples of such states, for both focusing ($\sigma = +1$) and defocusing ($\sigma = -1$) nonlinearities, shown on the band-gap diagram of the photonic lattice. In general, we identify two types of such nonlinear modes for each sign of the nonlinearity. These modes appear as being bifurcated from the edge points of the Brillouin zone, and they indeed resemble closely the flat-top discrete modes [4]. Moreover we find that, unlike the conventional 'narrow' discrete and gap solitons associated with a particular band of the band-gap spectrum of a periodic lattice [5, 14], such broad states may originate from any band edge becoming localized inside the gaps of the linear spectrum (see Fig. 1).

In order to obtain a deeper physical insight into the nature of these broad quasirectangular nonlinear states in photonic lattices, we consider the self-focusing case $\sigma = +1$, and we find numerically various families of such localized modes, which differ by the number of intensity peaks, as shown in Fig. 2a and b. Remarkably, these modes may be viewed in

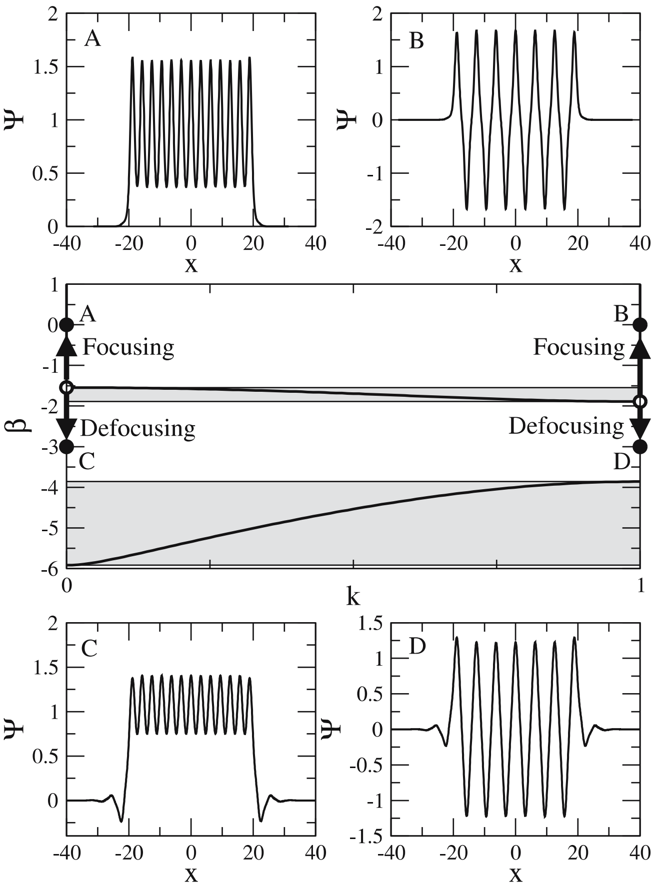

FIGURE 1 Examples of spatially localized broad nonlinear states in a periodic photonic lattice (at $V_0 = 4$). For the focusing nonlinearity ($\sigma = +1$) these states (A, B) exist in the semi-infinite upper band gap, while for the defocusing nonlinearity ($\sigma = -1$) they exist (C, D) inside the Bragg-reflection gap. *Center plot* shows the band-gap diagram

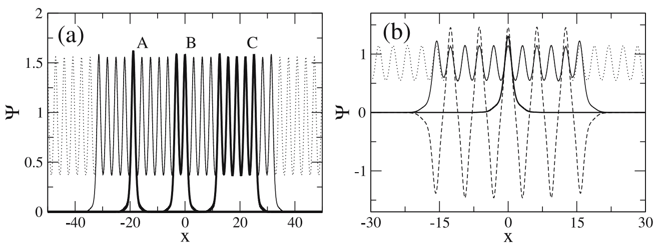

FIGURE 2 (**a**) Examples of one (A), two (B), and five (C) solitons and an extended localized state (*thin line*) superimposed on the nonlinear Bloch wave (*dotted line*) ($V_0 = 4$, $\beta = 0$). (**b**) Comparison of in-phase (*thin solid line*) and out-of-phase (*dashed line*) truncated states with a single soliton (*thick solid line*). *Dotted line* shows the nonlinear Bloch wave ($V_0 = 2$, $\beta = 0$)

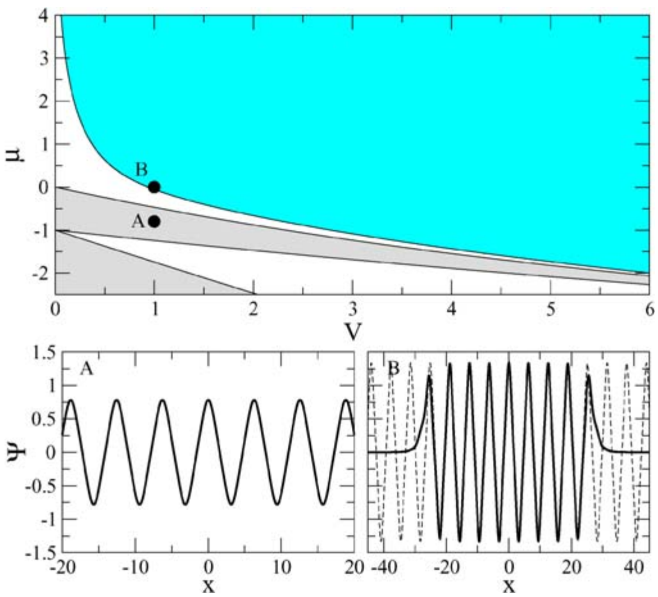

FIGURE 3 *Top:* existence region (*upper shaded*) and the band structure (*lower shaded*) vs. V_0 for the staggered truncated nonlinear Bloch waves. *Bottom:* A, an example of the staggered nonlinear Bloch wave inside the linear band ($\beta = -0.8$, $V_0 = 1$); B, truncated nonlinear Bloch wave near the cut-off (*solid line*) and the corresponding nonlinear Bloch wave (*dashed line*) ($\beta = 0$, $V_0 = 1$)

two distinct ways. Firstly, all such modes can be understood as 'soliton complexes', i.e. bound states of a particular number of isolated solitons placed in-phase or out-of-phase [15] at the minima of the lattice periodic potential and stabilized by the lattice. This is clearly seen when we compare simple states of one or two solitons with one of the broad states; see Fig. 2a. On the other hand, comparing these broad states with the corresponding nonlinear periodic solutions of Eq. (1), we may treat them as 'truncated nonlinear Bloch modes' existing inside the band gaps of a periodic potential (see Fig. 3).

In Fig. 2b we compare in-phase (thin solid line) and out-of-phase (dashed line) broad nonlinear states with a single soliton (thick solid line) and a nonlinear Bloch wave (dotted line) and obtain a direct confirmation of the nonlinear Bloch wave origin of the broad states. Indeed, comparing the maximum amplitudes of the two modes we see that the in-phase mode is lower as it is closer to its bifurcation point (see Fig. 1). This is contrary to the defocusing case, where it is the out-of-phase mode which is closest to cut-off and therefore lower in amplitude (see Fig. 1). In this language, a single soliton is then a borderline case between the two modes, and it sits between them in amplitude as is evident in Fig. 2b. As the potential deepens, the difference between the out-of-phase and in-phase modes becomes smaller as the band narrows, bringing the two bifurcation points closer together.

3 Generation of nonlinear modes

From the physical point of view the existence of the two types of modes, that in agreement with the discrete models can be called 'staggered' and 'unstaggered' [4], is quite clear since only the interaction of in-phase or out-of-phase solitons of equal amplitude can suppress the energy exchange between neighboring solitons [15], while the whole structure remains stationary due to the trapping potential of the lattice. However, when the periodic lattice is removed (see Fig. 4), the broad nonlinear mode develops a strong instability, and decays into a rapidly expanding 'soliton gas', as shown in Fig. 4 at $z = 40$.

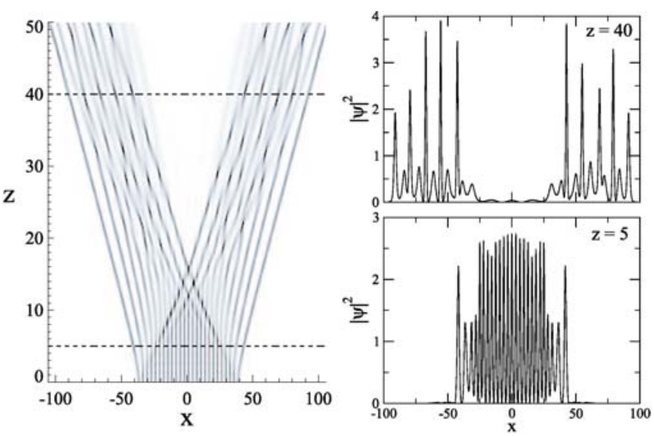

FIGURE 4 Propagation of the out-of-phase truncated nonlinear Bloch state when the periodic lattice is removed at $z = 0$, with strong repulsion and soliton-gas behavior. *Left:* evolution of the spatial intensity pattern. *Right:* examples of the field intensities at *dashed lines* on the *left-hand plot*. Initial state corresponds to the mode at $V_0 = 4$, $\beta = 0$

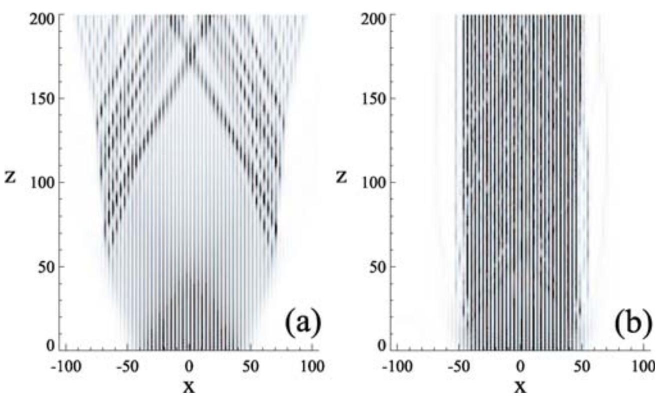

FIGURE 5 Evolution of a super-Gaussian beam with staggered phase, inside a lattice with $V_0 = 4$. (**a**) Diffraction due to sub-critical initial intensity of the super-Gaussian beam. (**b**) Self-trapping of a truncated nonlinear Bloch wave

Similar to the case of the discrete NLS equation, broad nonlinear modes may suffer from spatial modulational instability, and strong instability is indeed observed for the unstaggered modes in the case of the focusing nonlinearity, i.e. for the mode A in Fig. 1, similar to the discrete models [16]. However, we find the staggered modes (e.g. the mode B in Fig. 1) to be very robust during numerical propagation. Conversely, it is the unstaggered modes which are found to be robust in defocusing media. The analysis of the modulational instability and stability of these novel nonlinear modes is beyond the scope of this paper.

A natural question arises of how to generate these novel types of nonlinear modes in experiments. In order to model the generation of robust staggered modes, we use as input a phase-modulated broad super-Gaussian beam and study the beam evolution in a periodic lattice for increasing beam amplitudes. When the input power is weak, such a broad beam undergoes slow discrete diffraction, similar to the evolution of narrow beams. Below some threshold power, the diffraction becomes clearly nonlinear (see Fig. 5a). The broad beam becomes self-trapped above a threshold power, and it generates a stable broad state consisting of a sequence of many, almost identical optical solitons trapped by the lattice potential, as shown in Fig. 5b.

4 Links to the Bose–Einstein condensates

It is important to note that the NLS equation (1) with $\sigma = -1$ discussed above serves as the basic model for the study of the dynamics of cigar-shaped Bose–Einstein condensates trapped in an optical lattice. Therefore, in application to the theory of Bose–Einstein condensates, the novel broad nonlinear gap states with an arbitrary degree of localization we have introduced and described may serve, similar to the optical case discussed above, as a missing link between the spatially extended nonlinear Bloch waves [17, 18] and matter-wave gap solitons [18–20]. These states appear as 'truncated' nonlinear Bloch waves, and they are localized strictly in the gaps of the linear Bloch wave spectrum, and as such we shall refer to them as 'gap waves'.

The mechanism for this truncation intimately connects the broad-gap states to the nonlinearly self-trapped states, as has been recently experimentally reported by Anker et al. [13]. In that experiment, the increasing nonlinearity of the Bose–Einstein condensate wavepacket enabled the transition from the diffusive regime of condensate expansion in a deep one-dimensional optical lattice to the self-trapping regime, where the initial expansion stopped and the width remained finite. This self-trapping effect in tight-binding potentials was previously discussed in the context of discrete lattices [21]. Our analysis shows that, although the broad-gap states can be excited only above a certain density threshold, they exist well beyond the tight-binding regime self-trapping concept. Moreover, we have found similar nonlinear states in 1D, 2D, and 3D periodic potentials; these results will be reported elsewhere.

5 Conclusions

We have analyzed a continuous analog of the flat-top quasirectangular nonlinear modes earlier found for the discrete NLS equation. We have discussed that these modes can be understood as both truncated nonlinear Bloch waves and multi-soliton complexes with in-phase or out-of-phase neighboring solitons trapped by the periodic potential of a photonic lattice. Finally, we have demonstrated how to generate these novel types of self-trapped nonlinear modes in experiments.

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