Nonlinear Dynamics and Solitons in the Presence of Rapidly Varying Periodic Perturbations

YURI S. KIVSHAR
Optical Sciences Centre, Australian National University, Australian Capital Territory, 0200 Canberra, Australia

and

KARL H. SPATSCHEK
Institut für Theoretische Physik 1, Heinrich-Heine-Universität Düsseldorf, D-40225 Düsseldorf, Germany

(Received 6 October 1993; revised 30 November 1993)

Abstract — Problems of nonlinear dynamics and soliton propagation in the presence of rapidly varying periodic perturbations are considered applying a rigorous analytical approach based on asymptotic expansions. The method we develop allows derivation of an effective nonlinear equation for the slowly varying field component in any order of the asymptotic procedure as expansions in the parameter \( \alpha^{-1} \), \( \alpha \) being the frequency of the rapidly varying (direct or parametric) driving force. The general approach is demonstrated on several examples of different physical nature, including chaos suppression in the parametrically driven Duffing oscillator, dynamics of the sine–Gordon kinks in the presence of rapidly varying direct or parametric driving force, propagation of envelope (nonlinear Schrödinger) solitons in optical fibres with periodic amplification, stability of solitons on rapidly varying spatial periodic potential, and so on.

1. INTRODUCTION

As is well known, the effect of rapidly varying perturbations on the dynamics of strongly nonlinear systems can lead to a drastic change of the averaged system behaviour. In particular, large-amplitude parametric perturbations give rise to a stabilization of certain dynamical regimes. A famous example is the stabilization of a reverse pendulum by parametrically forced oscillations of its pivot [1]. A similar stabilization may be also achieved by a direct large amplitude driving force [2]. As has been shown recently, the dynamical stabilization of the pendulum has counterparts in nonlinear systems with many degrees of freedom, supporting, in particular, new types of kink solitons [3–6]. However, most methods usually used to derive an averaged equation for the nonlinear dynamics in the presence of rapidly varying perturbations are not rigorous. Such methods, even being very clear from the physical point of view, use a splitting into slow and fast variables. The subsequent averaging is based, in fact, on solutions of the linearized equations for fast variations where the slowly varying part is assumed to be constant (see, e.g. ref. [1]). The linearization assumes that the amplitudes of the forced (rapidly varying) oscillations are small, and this is certainly valid for small parametric perturbations far from parametric resonance. For direct, a.c. perturbations, the forced oscillations may become large. To describe the dynamics in an approximate way, the so-called ‘rotating-wave approximation’ was used without detailed justification [5, 6]. It is necessary to note that the derivation of an effective averaged equation for the slowly varying field component is an important
problem, and in many of the cases cases the corresponding equation determines the leading effects. In all the cases, it is necessary to justify the averaging procedure as well as to estimate the influence of the higher-order contributions. Unfortunately, the latter are beyond the usual averaging method. However, as we show in the present paper, a rigorous analysis of the effect of rapidly varying periodic perturbations on nonlinear dynamics can be performed in a straightforward manner.

The purpose of the present paper is to review various physical problems involving rapidly varying periodic perturbations to nonlinear and soliton-bearing models. We shall present all the results in the framework of a rigorous analytical approach based on asymptotic series expansions where the coefficients are assumed to be slowly varying functions on the timescale $\omega^{-1}$. Here $\omega$ is the frequency of the rapidly varying ac force which is assumed to be large. The basic idea to split into fast and slow variables is not new. However, the analytical method used in this paper to derive an effective equation for the slowly varying field component is to our knowledge novel. This method allows us to make, in a self-consistent way, all the corrections using solely asymptotic expansions rather than averaging in fast oscillations by making use of some physically plausible approximations.

The paper is organized as follows. In Section 2 we elucidate the main idea of this approach when analysing nonlinear oscillations of the Duffing oscillator driven by a high-frequency parametric force. One of the main problems considered there is the chaos suppression following from the renormalized dynamics described by the averaged equation. Section 3 is devoted to the sine–Gordon (SG) kink dynamics in the presence of direct and parametric driving forces. In particular, we show a qualitative difference between the two latter cases and, as a consequence, the very different physical phenomena caused by such rapidly oscillating forces. Another interesting physical example, showing how the renormalized dynamics may differ from the dynamics of the primary (unperturbed) model, is the prediction of the stable kinks on rotating and oscillating backgrounds. The concept of the average soliton in optical fibres with periodic (rapidly varying) amplification is discussed in Section 4. There it is shown that in the leading order of the asymptotic expansion the problem can be effectively analysed in the framework of a renormalized nonlinear Schrödinger (NLS) equation. The method developed here can be applied easily to rapidly varying spatial perturbations, and we demonstrate one of the examples in Section 5, considering the NLS solitons on a rapidly varying (sinusoidal) periodic potential. Finally, Section 6 summarizes the conclusions.

2. PARAMETRICALLY DRIVEN DUFFING OSCILLATOR

2.1. Preliminary remarks

The basic method can be exemplified on the (classical) Duffing oscillator. Let us consider the driven and damped Duffing oscillator

$$\frac{d^2x}{dt^2} - \alpha(t)x + \beta x^3 = -\gamma \frac{dx}{dt} + F \cos(\omega t), \quad (1)$$

with a parametric force taken, for simplicity, as

$$\alpha(t) = \alpha[1 + \epsilon \cos(\Omega t)]. \quad (2)$$

The frequency $\Omega$ of the 'parametric' force is assumed to be large in comparison with the frequency $\omega$ of the 'direct' driving force.

Considering the parametric force as rapidly oscillating, we separate the different time scales. According to [1], we should decompose the function $x(t)$ into a sum of slowly and
rapidly varying parts, i.e. \( x(t) = X(t) + \xi(t) \). The function \( \xi(t) \) stands for fast oscillations around the slowly varying envelope function \( X(t) \), and the mean value of \( \xi(t) \) during an oscillation period is assumed to be zero so that \( \langle x \rangle = X \). One of the simplest ways to obtain the renormalized equation for the slowly varying oscillations is to split equation (1) into two by averaging over the fast oscillations and subtracting (see for the basic idea ref. [1]). However, further simplifications are usually not properly justified, so that the so-called *mean-field approximation* does not give a rigorous analytical procedure to separate fast and slow variables. Below we formulate a procedure to solve this problem by means of asymptotic expansions.

### 2.2. Asymptotic expansions

The rapidly varying parametric force generates oscillations of large frequency \( \Omega \), so that we may look for the solution of equations (1) and (2) in the form of harmonic series,

\[
  x = X + \epsilon[A \cos(\Omega t) + B \sin(\Omega t) + C \cos(2\Omega t) + D \sin(2\Omega t) + \ldots].
\]

(3)

Here the parameter \( \epsilon \) is introduced for convenience, and the coefficients \( A, B, \ldots \) are assumed to be slowly varying on the timescale \( \sim \Omega^{-1} \). Substituting the expression (3) into (1) and (2), collecting the coefficients in front of the different harmonics, we obtain an infinite set of coupled nonlinear equations,

\[
  \frac{d^2X}{dt^2} - \alpha X + \beta X^3 + \frac{3}{8} \epsilon^2 \beta X(A^2 + B^2 + \ldots) - \frac{1}{2} \alpha \epsilon^2 A = -\gamma \frac{dX}{dt} + F \cos(\omega t),
\]

(4)

\[
  \left( -\Omega^2 A + \frac{dB}{dt} + \frac{d^2A}{dt^2} \right) - \alpha A + \gamma \left( \frac{dA}{dt} + \Omega B \right) + \beta \left( 3X^2 A + \frac{3}{2} \epsilon^2 A^3 \right) + \ldots = \frac{1}{2} \alpha C,
\]

(5)

\[
  \left( -\Omega^2 B - \frac{dA}{dt} + \frac{d^2B}{dt^2} \right) - \alpha B + \gamma \left( \frac{dB}{dt} - \Omega A \right) + \beta \left( 3X^2 B + \frac{3}{2} \epsilon^2 A^2 B + \ldots \right) = \frac{1}{2} \alpha A,
\]

(6)

\[
  \left( -4\Omega^2 C + 2\Omega \frac{dD}{dt} + \frac{d^2C}{dt^2} \right) - \alpha C + \gamma \left( \frac{dC}{dt} + 2\Omega D \right) + \beta \left( 3X^2 C + \frac{3}{2} \epsilon X A^2 \right) + \ldots = \frac{1}{2} A,
\]

(7)

\[
  \left( -4\Omega^2 D - 2\Omega \frac{dC}{dt} + \frac{d^2D}{dt^2} \right) - \alpha D + \gamma \left( \frac{dD}{dt} - 2\Omega C \right) + \beta \left( 3X^2 D + 3\epsilon X A B \right) + \ldots = \frac{1}{2} B,
\]

(8)

and so on. To proceed further, we note that equations (5–8) allow asymptotic expansions for the functions \( A, B, \ldots \). If the parameter \( \Omega \) is large, the term \( -\Omega^2 A \) in (5) may be compensated by the term \( \alpha X \) if one assumes \( A \sim \Omega^{-2} \). From (6), which has no perturbation-induced terms, it simply follows that the largest term to be compensated by \( -\Omega^2 B \) must be of order of \( \Omega A \). Such a simple consideration motivates the following asymptotic expansions:

\[
  A = \frac{a_1}{\Omega^2} + \frac{a_2}{\Omega^3} + \ldots \quad B = \frac{b_1}{\Omega^3} + \frac{b_2}{\Omega^4} + \ldots \quad C = \frac{c_1}{\Omega^4} + \ldots \quad D = \frac{d_1}{\Omega^5} + \ldots
\]

(9)

Substituting (9) into (5–8) and equating terms of the same orders in \( \Omega \), we find

\[
  a_1 = -\alpha X,
\]

(10)
\[
\frac{da_1}{dt} + \frac{d^2 a_1}{dt^2} - \alpha a_1 + 3 \beta X^2 a_1 + \gamma \left( \frac{da_1}{dt} + b_1 \right),
\]
(11)

\[
b_1 = -\frac{da_1}{dt} - \gamma a_1,
\]
(12)

\[
\frac{da_2}{dt} + \frac{d^2 a_2}{dt^2} - a b_1 + 3 \beta X^2 b_1 + \gamma \left( \frac{db_1}{dt} - a_2 \right),
\]
(13)

\[
c_1 = -\frac{1}{2} a_1, \quad d_1 = -\frac{1}{2} b_1,
\]
(14)

and so on. The parameter \( \delta = \epsilon / \Omega \) is assumed to be of the order \( O(1) \), but all the results are certainly valid also for \( \delta \ll 1 \). The expansions (9) allow to find the coefficients in each order of \( \Omega^{-1} \), and all the corrections are determined only by algebraic relations. For example, \( a_2 \) is given by (11) through \( b_1 \) which, in turn, may be found from (12) as a function of \( a_1 \), i.e. through the slowly varying component \( X \). This behaviour is typical for all coefficients of the asymptotic expansion.

Applying the expansion (9) to the equation (4) for the slowly varying oscillation component \( X \), we find the equation

\[
\frac{d^2 X}{dt^2} - \alpha X + \beta X^3 - \frac{1}{2} \alpha \delta^2 \left( a_1 + \frac{a_2}{\Omega^2} + \ldots \right) + \frac{3}{2} \beta \delta^2 X \left( \frac{a_1^3}{\Omega^2} + \frac{2 a_1 a_2 + b_1^2}{\Omega^4} + \ldots \right) =
\]

\[
-\gamma \frac{dX}{dt} + F \cos(\omega t).
\]
(15)

From here it is quite obvious how to get the first, second, and subsequent orders of the approximation to determine the averaged equation.

In the first-order approximation only the term \( -\delta a_1 \) contributes, so that (15) yields

\[
\frac{d^2 X}{dt^2} - \bar{\alpha} X + \beta X^3 = -\gamma \frac{dX}{dt} + F \cos(\omega t),
\]
(16)

where

\[
\bar{\alpha} = \alpha \left( 1 - \frac{1}{2} \alpha \delta^2 \right).
\]
(17)

Equations (16) and (17) take into account an effective contribution of the rapidly varying parametric force to the 'average' nonlinear dynamics and this contribution will become large when \( \delta = O(1) \). Thus, the dynamics of the Duffing oscillator with a rapidly varying parametric forcing can (to lowest order) be described by a renormalized Duffing equation (16), and the highest-order corrections to this equation are of order \( \epsilon^2 / \Omega^4 \). In fact, applying the expansions to get the corrections of the next order approximation, we can show that this result is still valid up to terms of order of \( O(\Omega^{-2}) \). The corresponding (renormalized) coefficients of the Duffing equation are

\[
\bar{\alpha} = \alpha \left[ 1 - \frac{1}{2} \alpha \delta^2 + \frac{\alpha \delta^2}{2 \Omega^2} (\alpha + \gamma^2) \right],
\]
(18)

\[
\bar{\beta} = \beta \left( 1 + \frac{3 \alpha^2 \delta^2}{\Omega^2} \right),
\]
(19)

\[
\bar{\gamma} = \gamma \left( 1 - \frac{\alpha^2 \delta^2}{2 \Omega^2} \right).
\]
(20)
2.3. Chaos suppression

As we have shown above, the averaged dynamics of the Duffing oscillator subjected to rapidly varying parametric perturbations may be described again by a Duffing equation but with renormalized coefficients. This conclusion is rather nontrivial, and it means that we can apply all the results known for the latter equation to analyse different kinds of system dynamics, including the regular and chaotic regimes. For example, the threshold of chaos, which is defined by the value of the a.c. driving force \( F \) producing the appearance of a strange attractor in the Poincaré sections, may be estimated by the classical Melnikov method [7] (see also [8, 9]). The method consists of evaluating the distance \( \Delta(t_0) \) between stable and unstable manifolds which, in the present cases, form homoclinic loops. In fact, in the presence of dissipation the homoclinic loop is destroyed, but it may be recovered by adding a force, provided that the force amplitude exceeds a certain critical value. To find the critical value, one should check if the function \( \Delta(t_0) \) changes its sign for some \( t_0 \).

The Melnikov function \( \Delta(t_0) \) is defined by the relation

\[
\Delta(t_0) = \int_{-\infty}^{\infty} dt \phi_{0}(t) \mathcal{R}[\phi_{0}(t), \dot{\phi}_{0}(t) + t + t_0],
\]

where \( \phi_{0}(t) \) is the homoclinic orbit evaluated in the absence of perturbations (i.e. without losses and driving), \( \mathcal{R} \) is the right-hand side of equation (16), and the dot stands for the time derivation. We should note, however, that the Melnikov method actually deals with the occurrence of transversal homoclinic points, but it does not characterize the global dynamics of the system, so that, in general, the actual threshold (observed in practice) becomes ‘visible’ a little bit above the Melnikov value (see, e.g. ref. [9] for more discussions of that point).

For the Duffing oscillator, the Melnikov function is (see, e.g. ref. [8])

\[
\Delta(t_0) = \pi \omega F \sqrt{\frac{2}{\beta}} \text{sech} \left( \frac{\pi \omega}{2\sqrt{\tilde{\alpha}}} \right) \sin (\omega t_0) + \frac{4 \gamma \tilde{\alpha}^{3/2}}{3 \beta},
\]

and the condition to prevent \( \Delta(t_0) \) from changing the sign takes the form:

\[
\gamma > \frac{3\pi F \sqrt{\beta \omega}}{(2\tilde{\alpha})^{3/2}} \text{sech} \left( \frac{\pi \omega}{2\sqrt{\tilde{\alpha}}} \right).
\]

Crossing of stable and unstable manifolds, as is determined by the Melnikov function, gives only the criterion for the onset of chaotic motion in the limit of low dissipation when the transient times are much longer than the characteristic timescale of the system dynamics. The critical values of the parameters allowing a chaotic motion are roughly given by the criterion that the averaged double-well potential changes to a single-well potential (see, e.g. ref. [10] and discussions therein). This condition yields the critical dependence \( \varepsilon \approx \sqrt{2}\Omega \) which separates chaotic and regular motion for the averaged dynamics.

To support the idea of the chaos suppression formulated above, numerical simulations of the system described by equations (1) and (2) have been performed in ref. [11] to extract numerically the Lyapunov characteristic exponent. The relevant Lyapunov exponent versus \( \varepsilon \) is shown in Fig. 1. The important conclusion which follows from such dependencies is twofold. First, the Lyapunov exponent vanishes for large values of \( \varepsilon \) and thus a regular motion is actually recovered. Second, a set of ‘windows’, where the Lyapunov exponent is sufficiently suppressed, or even becomes negative, is observed in the simulation at small \( \varepsilon \). Figures 1 (b–f) display how the oscillations become regular. More detailed discussion of the chaos suppression and the validity of the averaged equation (16) may be found in ref. [11].
Fig. 1. Numerical simulation of equation (1) for $F = 0.35$, $\gamma = 0.4$, $\Omega = 10.0$, and $\omega = \alpha = \beta = 1.0$. (a) Lyapunov exponent vs. amplitude of the parametric force $\epsilon$. The negative sign of the exponent for $\epsilon > 10^{1/2}$ indicates suppression of chaos. Note the occurrence of periodic 'windows' at lower $\epsilon$; (b) and (c) time-dependence for the directly forced oscillator ($\epsilon = 0$) and (e) for the stabilized case ($\epsilon = 15$), respectively; (d) and (f) corresponding power spectra, respectively. Time, frequency and Lyapunov exponents are measured in terms of a system-time unit.

However, we should note that at very large values of $\epsilon$ the regularized averaged dynamics may become chaotic again. Indeed, as follows from equation (18), which determines the most critical parameter for the threshold of chaos, the higher-order correction acts with the sign ‘+’, so that it may help to recover the chaotic dynamics again.
due to the driving effect of the h.f. parametric modulation itself. As has been checked numerically in [11], this occurs at $\Omega = 1.0$ for $\epsilon > 0.9$, at $\Omega = 3$ for $\epsilon > 13$, at $\Omega = 5.0$ for $\epsilon > 33$, and so on. Nevertheless, the ‘windows’ where the chaos is suppressed or completely eliminated are wide enough to be of practical importance.

To conclude this section, we would like to mention that recently considerable work has been done in controlling chaos by periodic parametric perturbations (including the Duffing oscillator) using both analytical and numerical technique (see, e.g. refs. [12-14] to cite a few). However, the suppression of chaos discussed above is different from stabilization by mode-locking where $\Omega$ in equations (1), (2) should be near a small multiple of $\omega$. In this latter case the suppression is seen to come about by an increase of the length of the laminar periods. This is very different from the chaos suppression discussed above: The fast modulations lead to a new stable trajectory of the averaged dynamical system.

3. SINE-GORDON KINKS IN THE PRESENCE OF PERIODIC FORCES

3.1. Direct driving force

First, we consider the case of the direct driving force in the SG model when the system dynamics is described by the equation

$$\frac{\partial^2 \Phi}{\partial t^2} - \frac{\partial^2 \Phi}{\partial x^2} + \sin \Phi = f - \gamma \frac{\partial \Phi}{\partial t} + \epsilon \cos(\omega t),$$

(24)

where $\gamma$ is assumed to be of order of 1, and the amplitude $\epsilon$ of the driving force may be large (in fact, up to the values $\epsilon \sim \omega^2$). In the subsequent analysis we consider the direct driving force as rapidly oscillating, i.e. the external frequency $\omega$ is assumed to be large in comparison with the linear frequency gap ($= 1$). Our purpose is to derive an averaged nonlinear equation to describe the slowly varying dynamics of the SG field.

In order to derive an averaged equation of motion, we decompose the field $\Phi$ into a sum of slowly and rapidly varying parts, i.e.

$$\Phi = \Phi + \zeta,$$

(25)

and look for the rapidly oscillating function $\zeta$ in the form [cf. (3)]

$$\zeta = A \cos(\omega t) + B \sin(\omega t) + C \cos(2\omega t) + D \sin(2\omega t) + \ldots,$$

(26)

where the coefficients $A, B, \ldots$ are slowly varying on the time scale $\sim \omega^{-1}$. Substituting the expressions (25 and 26) into (24), and collecting the coefficients in front of the different harmonics, we obtain an infinite set of coupled nonlinear equations,

$$\frac{\partial^2 \Phi}{\partial t^2} - \frac{\partial^2 \Phi}{\partial x^2} + \sin \Phi[J_0(A)(1 - \frac{1}{4}B^2) + \frac{1}{2}B^2J_2(A) + \ldots] +$$

$$\cos \Phi[-CJ_2(A) + \ldots] = f - \gamma \frac{\partial \Phi}{\partial t},$$

(27)

$$\left(-\omega^2 A + 2\omega \frac{\partial B}{\partial t} + \frac{\partial^2 A}{\partial t^2}\right) - \frac{\partial^2 A}{\partial x^2} + \cos \Phi[2J_1(A) + \ldots] +$$

$$\sin \Phi[-CJ_2(A) + \ldots] + \gamma \left(\frac{\partial A}{\partial t} + \omega B\right) = \epsilon,$$

(28)
\[
\left( -\omega^2 B - 2\omega \frac{\partial A}{\partial t} + \frac{\partial^2 B}{\partial t^2} \right) - \frac{\partial^2 B}{\partial x^2} + \cos \Phi [BJ_0(A) + \ldots] + \\
\sin \Phi [-D J_1(A) + \ldots] + \gamma \left( \frac{\partial B}{\partial t} - \omega A \right) = 0, \tag{29}
\]
\[
\left( -4\omega^2 C + 4\omega \frac{\partial D}{\partial t} + \frac{\partial^2 C}{\partial t^2} \right) - \frac{\partial^2 C}{\partial x^2} + \cos \Phi [CJ_0(A) + \ldots] + \\
\sin \Phi \left[ \frac{1}{2} B^2 J_0(A) + \ldots \right] + \gamma \left( \frac{\partial C}{\partial t} + 2\omega D \right) = 0, \tag{30}
\]
\[
\left( -4\omega^2 D - 4\omega \frac{\partial C}{\partial t} + \frac{\partial^2 D}{\partial t^2} \right) - \frac{\partial^2 D}{\partial x^2} + \cos \Phi [DJ_0(A) + \ldots] + \\
\sin \Phi [-B J_1(A) + \ldots] + \gamma \left( \frac{\partial D}{\partial t} - 2\omega C \right) = 0, \tag{31}
\]
and so on. To proceed further, we note that equations (28–31) allow application of asymptotic expansion method. If the parameter \( \omega \) is large, we may expand the coefficients \( A, B, \ldots \) in terms of \( \omega^{-1} \) as follows [cf. (9)]
\[
A = a_1 + \frac{a_2}{\omega^2} + \ldots, \quad B = \frac{b_1}{\omega} + \frac{b_2}{\omega^3} + \ldots, \quad C = \frac{c_1}{\omega^4} + \ldots, \quad D = \frac{d_1}{\omega^3} + \ldots \tag{32}
\]
Substituting (32) into (28–31) and equating terms of the same orders in \( \omega^{-1} \), we find
\[
a_1 = -\frac{\epsilon}{\omega^2} = -\delta, \tag{33}
\]
\[
a_2 = \gamma b_1 + 2 \cos \Phi J_1(a_1), \tag{34}
\]
\[
b_1 = -\gamma a_1, \tag{35}
\]
\[
b_2 = -2 \frac{\partial a_2}{\partial t} - \gamma a_2 + b_1 \cos \Phi [J_0(a_1) + 2J_2(a_1)], \tag{36}
\]
\[
c_1 = \frac{1}{4} \left[ 4 \frac{\partial d_1}{\partial t} + \frac{1}{2} b_1^2 J_0(a_1) \sin \Phi + \frac{1}{2} b_1^2 J_2(a_1) + 2 \gamma d_1 \right], \tag{37}
\]
\[
d_1 = -\frac{1}{2} b_1 J_1(a_1) \sin \Phi, \tag{38}
\]
and so on. In all the equations written above we did not take into account the derivatives of \( a_1 \) and \( b_1 \) because \( a_1 \) and \( b_1 \sim a_1 \) are constant as it follows from the first terms of the asymptotic expansions. In equation (33) the parameter \( \delta = \epsilon/\omega^2 \) is assumed to be of order of \( O(1) \).

Applying the expansions (32) to (27), we can find the equation for the slowly varying field component \( \Phi \) with any accuracy in the small parameter \( \omega^{-1} \),
\[
\frac{\partial^2 \Phi}{\partial t^2} - \frac{\partial^2 \Phi}{\partial x^2} + \sin \Phi \left\{ J_0(a_1) + \frac{1}{\omega^2} \left[ -\frac{1}{2} b_1^2 J_0(a_1) - a_2 J_1(a_1) + \frac{1}{2} b_2^2 J_2(a_1) \right] + \ldots \right\} \\
+ \cos \Phi \left\{ -\frac{c_1}{\omega^4} J_2(a_1) + \ldots \right\} = f - \gamma \frac{\partial \Phi}{\partial t}. \tag{39}
\]
In the first-order approximation only the term \( J_0(a_1) \sin \Phi \) contributes, so that (39) reads
\[
\frac{\partial^2 \Phi}{\partial t^2} - \frac{\partial^2 \Phi}{\partial x^2} + J_0 \left( \frac{\epsilon}{\omega^2} \right) \sin \Phi = f - \gamma \frac{\partial \Phi}{\partial t}. \tag{40}
\]
Equation (40) takes into account an effective contribution of the rapidly varying force to the average nonlinear dynamics, and this contribution can become large for \( \delta = \mathcal{O}(1) \), i.e. when \( \epsilon \sim \omega^2 \). Thus, the dynamics of the SG model with a rapidly varying direct force may be described by a renormalized SG equation (40) up to the order \( \mathcal{O}(\epsilon/\omega^2) \).

The results obtained above can be immediately applied to describe the dynamics of kinks in the presence of the rapidly varying a.c. force. In fact, equation (40) is the d.c.-driven damped SG equation with a renormalized coefficient in front of the term \( \sim \sin \Phi \). This simply means that we can apply all the results known for the standard SG equation (see, e.g. ref. [15]) by introducing only the renormalization of the kink’s width. For example, the kink solution of equation (40) at \( \gamma = f = 0 \) has the form

\[
\Phi_k(x, t) = 4\sigma \tan^{-1} \exp \left[ \frac{x - V_t}{l_0 \sqrt{1 - V^2}} \right],
\]

where \( \sigma = \pm 1 \) is the kink’s polarity and \( l_0 = J_0(\epsilon/\omega^2) \) is the kink’s width at rest. The motion of the kink in the presence of small d.c. force \( f \) and damping (\( \sim \gamma \)) is characterized by the steady-state velocity

\[
V_* = -\frac{\sigma}{\sqrt{1 + g^2}}, \quad g = \left( \frac{4\gamma}{\pi f} \right) J_0 \left( \frac{\epsilon}{\omega^2} \right).
\]

In the theory of long Josephson junctions kink’s velocity is connected with the voltage across the junction and the result (42) for the steady-state kink velocity gives the so-called zero-field steps in the current–voltage characteristics of a long junction. As follows from (42), the renormalization of the parameter \( g \) will lead to a change of the kink’s velocity \( V_*(f) \) and, therefore, this will lead to a change of slopes of the voltage steps in the presence of the a.c. driving force [16].

It is also interesting to note that the Bessel function \( J_0(x) \) may change its sign for larger values of the parameter \( \epsilon/\omega^2 \), so that the stable ground state \( \Phi = 0 \) becomes unstable. The so-called inverted ground state \( \Phi = \pm \pi \) may support the inverted kinks [5] with the properties similar to those of the standard SG kinks.

3.2. Parametric driving force

Let us consider now the case of a parametric driving force to demonstrate that such a situation is very different from that analysed above. The main qualitative difference of the effects produced by direct and parametric (rapidly oscillating) forces is the following. The sufficient change of the system dynamics due to a direct force is observed for the amplitudes \( \epsilon \sim \omega^2 \), whereas in the case of a parametric force the similar effects may be already observed for smaller amplitude, i.e. for \( \epsilon \sim \omega \). To prove the statement formulated above and to show how our asymptotic method works for the case of the parametric force, we consider the parametrically perturbed SG equation

\[
\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} + \sin \phi = f - \gamma \frac{\partial \phi}{\partial t} + \epsilon \sin \phi \cos (\omega t).
\]

We assume that the parametric force is rapidly oscillating, i.e. the frequency \( \omega \) is large. As above, we look for a solution of equation (43) in the form

\[
\phi = \Phi + A \cos (\omega t) + B \sin (\omega t) + C \cos (2\omega t) + D \sin (2\omega t) + \ldots,
\]

where the functions \( \Phi, A, B, \ldots \) are assumed to be slowly varying. The function \( \Phi \) in (44) is the ‘averaged’ field component because \( \langle \phi \rangle = \Phi \). Substituting the expression (44) into (43) and collecting the coefficients in front of the different harmonics, we obtain an infinite
set of coupled nonlinear equations. The subsequent (and very important) step of the analysis is to find the form of the asymptotic expansions for the coefficients $A$, $B$, $\ldots$. In the present case it is easy to check that the expansions (32) do not give a closed asymptotic procedure because the driving term ($\sim \varepsilon$) from (43) contributes to all the harmonics provided $\varepsilon \sim \omega^2$, so that contributions of the other harmonics are not small. However, for smaller amplitudes, i.e. when $\varepsilon \sim \omega$, the asymptotic procedure will give all the corrections to the averaged nonlinear dynamics in a rigorous way provided we take the expansions in the form [cf. (32)]

$$\begin{align*}
A &= \frac{a_1}{\omega^2} + \frac{a_2}{\omega^4} + \ldots, \\
B &= \frac{b_1}{\omega^3} + \frac{b_2}{\omega^5} + \ldots, \\
C &= \frac{c_1}{\omega^4} + \ldots, \\
D &= \frac{d_1}{\omega^5} + \ldots.
\end{align*}$$  

(45)

Using (45) we obtain

$$\frac{\partial^2 \Phi}{\partial t^2} - \frac{\partial^2 \Phi}{\partial x^2} + \left(1 - \frac{a_1^2}{\omega^4} + \ldots\right) \sin \Phi = f - \gamma \frac{\partial \Phi}{\partial t} + \varepsilon \cos \Phi \left(\frac{a_1}{\omega^2} + \frac{a_2}{\omega^4} + \ldots\right),$$  

(46)

where the parameter $\varepsilon/\omega$ is assumed to be of order of $\mathcal{O}(1)$, but all the results are valid also for $\varepsilon \ll \omega$. The expansions (45) allow for the coefficients in each order of $\omega^{-1}$, and also all the corrections are determined by algebraic relations.

In the first-order approximation only the term $\sim \varepsilon a_1$ contributes to (46), and from the asymptotic expansions it follows that

$$a_1 = -\varepsilon \sin \Phi.$$  

(47)

Thus, (46) transforms into the so-called double SG equation,

$$\frac{\partial^2 \Phi}{\partial t^2} - \frac{\partial^2 \Phi}{\partial x^2} + \left(1 + \frac{1}{2} \Delta^2 \cos \Phi\right) \sin \Phi = f - \gamma \frac{\partial \Phi}{\partial t},$$  

(48)

where $\Delta = \varepsilon/\omega$. Equation (48) takes into account an effective contribution of the rapidly varying parametric force to the average nonlinear dynamics in the lowest order, and all the corrections coming from the next-order terms are proportional to the smallness parameter $\omega^{-2}$. However, even the lowest-order contribution might become large for $\Delta = \mathcal{O}(1)$, i.e. when $\varepsilon \sim \omega$.

Thus, the dynamics of the SG system with a rapidly varying parametric force can be described by a double SG equation (48). As a matter of fact, the double SG equation is rather well studied (see, e.g. refs. [17, 18] and references therein), and properties of its kink solutions are known. In particular, the $2\pi$-kink solution of Eq. (48) at $f = \gamma = 0$ can be written in the form [17]

$$\Phi(x, t) = 2 \tan^{-1} \left[ \frac{1}{\sqrt{1 + \Delta^2/2}} \cosech \left( \sqrt{\left(1 + \frac{\Delta^2}{2}\right)} \frac{x - V_t t}{\sqrt{1 - V_t^2}} \right) \right],$$  

(49)

and for large $\Delta$ this solution may be treated as two coupled $\pi$-kinks. As has been shown in ref. [5], $\pi$-kinks themselves may exist in the parametrically driven SG chain provided the condition $\Delta^2 > 2$ is satisfied. This condition simply means that the effective 'averaged' potential for the slowly varying field component $\Phi$ exhibits a local minimum at $\Phi = \pi$ so that this stationary state becomes stable, and $\pi$-kink connects two stable stationary states $\Phi = 0$ and $\Phi = \pi$. This is exactly the same condition which appears in the problem of the parametric stabilization of the inverted pendulum [1].
3.3. Kinks on rotating and oscillating backgrounds

In ref. [6], it was shown that a rapidly varying force in the form of a rotating and oscillating background may drastically change the kink dynamics. Then the high-frequency force phase-locks the SG field in an oscillating and rotating state and thereby creates a mechanism (an effective gravitation field) for supporting kink solitons. To prove such a possibility analytically in a rigorous manner, we consider the perturbed SG equation (24) assuming \( f > 1 \) (so that the ground states \( \phi = 2\pi n \ (n = 0, \pm 1, \ldots) \) of the SG chain become unstable). The force supports a rotating state with the mean frequency \( \Omega \), so that \( \phi = \Omega t \). In the presence of high-frequency a.c. force \( \sim \epsilon \) to such a rotating state, we are interested in the slowly varying phase-locked system dynamics. Thus we anticipate a solution of equation (24) in the form

\[
\phi = \Phi + \Omega t + \xi,
\]

where \( \xi \) is the rapidly varying part oscillating with the frequency \( \omega \), \( \Phi \) is the slowly varying (long timescale) part, and \( \Omega \) is the average frequency of rotation for the background field. The latter we assume to be phase-locked to the external a.c. field, i.e. \( \Omega = \pm k\omega \), \( k \) being integer. Writing, as above, the rapidly oscillating part \( \xi \) as a Fourier series with slowly varying coefficients,

\[
\xi = A \cos(\omega t) + B \sin(\omega t) + \ldots,
\]

we obtain the following equations for the averaged field component \( \Phi \) and the coefficients \( A, B, \ldots \).

\[
\frac{\partial^2 \Phi}{\partial t^2} - \frac{\partial^2 \Phi}{\partial x^2} + J_k(A)\sin \Phi = f - \gamma k\omega - \gamma \frac{\partial \Phi}{\partial t},
\]

\[
\left(-\omega^2 A - 2\omega \frac{\partial B}{\partial t} + \frac{\partial^2 A}{\partial t^2}\right) - \frac{\partial^2 A}{\partial x^2} + [J_{k+1}(A) - J_{k-1}(A)]\cos \Phi + \gamma \left(\frac{\partial A}{\partial t} + \omega B\right) = \epsilon,
\]

\[
\left(-\omega^2 B + 2\omega \frac{\partial A}{\partial t} + \frac{\partial^2 B}{\partial t^2}\right) - \frac{\partial^2 B}{\partial x^2} - [J_{k+1}(A) + J_{k-1}(A)]\sin \Phi + \gamma \left(\frac{\partial B}{\partial t} - \omega A\right) = 0,
\]

and so on. Unlike the case considered above, in the present problem there are two rapidly oscillating contributions with the frequencies \( \omega \) and \( k\omega \), so that the final equations (52–54) for the slowly varying coefficients differ from the corresponding equations (27–29). To keep the second term on the right-hand side of equation (52) in a self-consistent way, we also assume the dissipation to be small, i.e. \( \gamma \omega \sim 1 \).

Using now the asymptotic expansions, i.e.

\[
A = a_1 + \frac{a_2}{\omega^2} + \ldots, \quad B = \frac{b_1}{\omega} + \ldots,
\]

we find the following results.

\[
a_1 = -\frac{\epsilon}{\omega^2} = -\delta, \quad b_1 = -\gamma a_1,
\]

which allow us to obtain the effective equation for the slowly varying dynamics just by combining equations (55, 56 and 52).

As has been mentioned above, numerical analysis of the kink propagation on a rotating and oscillating background was performed in ref. [6] and in Fig. 2 we show just one of the
examples when the stable propagation of a single kink on a rotating background is supported by a rapidly varying ac external force in the regime of the phase-locking. The numerical results agree with the predictions of equations (52).

4. RAPIDLY AMPLIFIED SOLITONS IN OPTICAL FIBRES

4.1. Asymptotic expansions

The approach developed above can be successfully applied [19] to investigate other types of solitons in the presence of rapidly varying periodic perturbations, e.g. envelope solitons described by the NLS equation. Let us consider an electric (complex) field envelope $u(t, z)$ in a fibre with periodically varying gain $if(z)u$ to compensate for the loss of a rate $\gamma$. To the lowest order in dispersion and nonlinearity, the envelope function $u(t, z)$ satisfies the well-known NLS equation (see, e.g. refs. [20, 21]),

$$\frac{i}{\partial z} - \frac{1}{z} \frac{\partial^2 u}{\partial t^2} + |u|^2 u = -i \gamma u + if(z)u. \tag{57}$$

Here the retarded time $t$ is normalized to the pulse width, the distance $z$ along a fibre is normalized to the soliton period, and the pulse amplitude $|u|$ is normalized to the soliton amplitude; the normalization units are well-known (see, e.g. ref. [20, 21]). Without restrictions of generality, we assume that the gain $f(z)$ is chosen to compensate for the losses on the average, i.e. $\gamma = \langle f(z) \rangle$, where the brackets stand for the averaging with respect to the fast variations. Introducing the new function with zero mean value, $\epsilon(z) = f(z) - \gamma$, we rewrite equation (57) in the following form,

$$\frac{i}{\partial z} + \frac{1}{z} \frac{\partial^2 u}{\partial t^2} + |u|^2 u = i \epsilon(z)u. \tag{58}$$

We consider first the simple case when the amplification function is harmonic, i.e. $\epsilon(z) = \epsilon \cos(qz)$. In this case the solution can be found by asymptotic expansions in harmonics of the rapidly varying amplification. In the general case, the corresponding solution is presented in the form of Fourier series, combining the coefficients found for the harmonic amplification (see below).

Fig. 2. The steady-state propagation of a kink on a rotating background shown for the field profile as a function of space over 10 periods of the external driving force. Parameters are: $\gamma = 1.0$, $\epsilon = 500$, $\Omega = 12.9$, and $f = 2.7$ [6].
Let us look for the rapidly varying part of the solution \( u = U + \xi \) in the form
\[
\xi = A \sin(qz) + B \cos(qz) + C \sin(2qz) + D \cos(2qz) + \ldots,
\]
where the coefficients \( A, B, \ldots \) are assumed to be slowly varying on the spatial scale \( q^{-1} \). Substituting the expression (59) into (58), and collecting the coefficients in front of the different harmonics, we obtain an infinite set of coupled nonlinear equations,
\[
i \frac{\partial U}{\partial z} + \frac{1}{2} \frac{\partial^2 U}{\partial t^2} + |U|^2 U + U(|A|^2 + |B|^2 + |D|^2) - \frac{1}{2} |A|^2 D - \frac{1}{4} A^2 D^* + \frac{1}{2} U^*(A^2 + B^2 + D^2) + \ldots = \frac{1}{2} i \epsilon B. \tag{60}
\]
\[
i \left( \frac{\partial A}{\partial z} - qB \right) + \frac{1}{2} \frac{\partial^2 A}{\partial t^2} + U^2 A^* + 2 |U|^2 A + \frac{3}{4} |A|^2 A + \frac{1}{2} A^2 D^* - UA^* D - ADU^* + \ldots = \frac{1}{2} i \epsilon C, \tag{61}
\]
\[
i \left( \frac{\partial B}{\partial z} + qA \right) + \frac{1}{2} \frac{\partial^2 B}{\partial t^2} + U^2 B^* + \frac{1}{2} A^2 B^* + 2 |U|^2 B + UAC^* + UBD^* + UCA^* + UDB^* + \frac{3}{4} |A|^2 B + ACU^* + BDU^* + \ldots = i \epsilon (U + \frac{1}{2} D), \tag{62}
\]
\[
i \left( \frac{\partial C}{\partial z} - 2qD \right) + \frac{1}{2} \frac{\partial^2 C}{\partial t^2} + U^2 C^* + 2 |U|^2 C + ABD^* + UAB^* + UA^* B + \ldots = \frac{1}{2} i \epsilon A, \tag{63}
\]
\[
i \left( \frac{\partial D}{\partial z} + 2qC \right) + \frac{1}{2} \frac{\partial^2 D}{\partial t^2} + U^2 D^* - \frac{1}{2} A^2 U^* + \frac{1}{2} A^2 D^* + \frac{1}{2} B^2 U^* - U|A|^2 + U|B|^2 + 2 |U|^2 D + |A|^2 D + \ldots = \frac{1}{2} i \epsilon B \tag{64}
\]
and similar equations for the coefficients of the higher-order harmonics. In equations (60–64) we display only the terms which in the subsequent expansions give contributions not smaller than those of order of \( q^{-4} \).

To proceed further, we note that (61–64) allow an asymptotic expansion method for the functions \( A, B, \ldots \) provided the parameter \( q \) characterizing the rapidly varying gain is assumed to be large. Then we expand \( A, B, \ldots \) in terms of \( q^{-1} \). It is easy to show that for the case \( \epsilon \sim O(1) \) the asymptotic expansions may be taken in the forms
\[
A = \frac{a_1}{q} + \frac{a_2}{q^2} + \ldots, \quad B = \frac{b_1}{q^2} + \frac{b_2}{q^4} + \ldots,
\]
\[
C = \frac{c_1}{q^3} + \frac{c_2}{q^5} + \ldots, \quad D = \frac{d_1}{q^3} + \frac{d_2}{q^5} + \ldots \tag{65}
\]

Substituting (65) into (61–64) and equating the terms of the same orders in \( q \), we find
\[
a_1 = \epsilon U, \tag{66}
\]
\[
a_2 = - \frac{\partial b_1}{\partial z} + \frac{1}{2} \epsilon d_1 + i \left[ \frac{1}{2} \frac{\partial^2 b_1}{\partial t^2} + U^2 b_1^* + 2 |U|^2 b_1 \right], \tag{67}
\]
\[
b_1 = \frac{\partial a_1}{\partial z} - i \left[ \frac{1}{2} \frac{\partial^2 a_1}{\partial t^2} + U^2 a_1^* + 2 |U|^2 a_1 \right], \tag{68}
\]
\[
b_2 = \frac{\partial a_2}{\partial z} - \frac{1}{2} \epsilon c_1 - i \left[ \frac{1}{2} \frac{\partial^2 a_2}{\partial t^2} + U^2 a_2^* + 2 |U|^2 a_2 + \frac{3}{4} |a_1|^2 a_1 - a_1 d_1^* U - a_1^* d_1 U - a_1 d_1 U * \right], \tag{69}
\]
c_i = -\frac{1}{2}\frac{\partial^2 d_i}{\partial z^2} + \frac{1}{4}\varepsilon b_1 + \frac{i}{2}\left[\frac{U^*}{2} + U^*(2Ud_1 - \frac{1}{2}a_1^2) - U|a_1|^2 + U^2d_i^2\right], \quad (70)

d_i = -\frac{1}{2}\varepsilon a_1, \quad (71)

etc. The expansions (65) allow us to find the coefficients in each order of \( q^{-1} \), and, as above, all the corrections are determined by algebraic relations.

4.2. Renormalized NLS equation

Using the expansions (65) in the averaged equation (60), we find the result

\[
i\frac{\partial U}{\partial z} + \frac{1}{2}\frac{\partial^2 U}{\partial t^2} + |U|^2U + \frac{1}{q^2}\left(\frac{|a_1|^2}{q^2} + \frac{a_1a_1^* + a_2a_2^*}{q^4} + \frac{|b|^2}{q^4} + \frac{|d|^2}{q^4} + \ldots\right) \\
+ \frac{1}{2}U^*\left(\frac{a_1^2}{q^2} + \frac{2a_1a_2}{q^4} + \frac{b_1^2}{q^4} + \frac{d_1^2}{q^4} + \ldots\right) - \frac{1}{2}|a_1|^2d_1 = \frac{1}{2}\varepsilon\left(\frac{b_1}{q^2} + \frac{b_2}{q^4} + \ldots\right). \quad (72)
\]

To find the first-order corrections in the NLS-like equation (72), we calculate the coefficients proportional to \( \varepsilon^2/q^2 \). The resulting equation for the slowly varying field component is found to be of the form

\[
(1 - \frac{1}{2}\delta)(i\frac{\partial U}{\partial z} + \frac{1}{2}\frac{\partial^2 U}{\partial t^2}) + |U|^2U + \mathcal{O}\left(\frac{\varepsilon^2}{q^4}\right) = 0, \quad (73)
\]

where \( \delta = \varepsilon/q \). Equation (73) does take into account an effective contribution of the rapidly varying gain to the ‘averaged’ nonlinear dynamics in the first-order approximation. It is important to note that all additional terms proportional to \( |U|^2U \) cancel, but the final equation (73) is again an NLS equation. Thus, the dynamics of optical solitons in fibres in the presence of losses and a rapidly varying gain may be described by a renormalized NLS equation (73) up to the order \( \varepsilon^2/q^4 \). We would like to point out that, although equation (73) is an NLS equation, it is not the same as the primary NLS equation without the amplification. The dynamics of the input pulses will differ from that in the non-amplified case. But note that (at least, on the distances less than \( q^4/\varepsilon^2 \)) for the averaged dynamics we have still an exactly integrable model.

One of the main assumptions used so far for the asymptotic expansions is the harmonic variation of the gain function. In general, a periodic function with zero mean can be presented in the form

\[
\varepsilon(z) = \sum_{n=1}^{\infty} [F_n \cos(qnz) + G_n \sin(qnz)],
\]

and, after similar calculations as above, we obtain the averaged equation

\[
\lambda^2\left(i\frac{\partial U}{\partial z} + \frac{1}{2}\frac{\partial^2 U}{\partial t^2}\right) + |U|^2U = 0, \quad (74)
\]

where now

\[
\lambda^2 = 1 - \frac{\varepsilon^2}{2q^2} \sum_{n=1}^{\infty} \frac{(F_n^2 + G_n^2)}{n^2}, \quad (75)
\]

This result generalizes equation (73) in a straightforward manner.

The renormalized NLS equations (73) or (74) follow here in the simplest way, together with the explicit expression for the second- (and higher-) order corrections. In the approach
based on the Lie transformation [22], a key point is to use a nonlinear transformation which makes an explicit evaluation rather tortuous and thus that approach is usually limited by the zeroth order which corresponds, in fact, to a direct averaging of the primary NLS equation. In the present section we have shown that for \( \epsilon \sim 1 \) and \( \epsilon/q \ll 1 \) the NLS equation is still valid up to the orders \( \epsilon^2/q^4 \), i.e. for the distances \( \sim q^4/\epsilon^2 \), but in its renormalized version.

4.3. Pulse propagation

A renormalization of the coefficients in equation (73) will lead to a physically important difference between the soliton dynamics for the renormalized and primary models, so that we can predict new features in the soliton propagation. One of the main effects, which might be clearly identified in practice, is expected for the propagation of a fundamental soliton in the presence of the periodic gain when the input pulse is taken in the form

\[
  u(t, 0) = \frac{b}{\cosh(bt)}.
\]

As follows from the standard consideration based on the inverse scattering transform [23], the input pulse (76) corresponds exactly to a soliton of the NLS equation without loss and gain (i.e. to equation (58) for \( \epsilon(z) = 0 \)), and the soliton has the form \( u(t, z) = b \text{sech}(bt) \exp(\frac{i}{2}b^2z) \). When propagating in a fibre with loss and periodic gain, the averaged dynamics of the input pulse (76) is now described by the renormalized NLS equation (73) and, therefore, the input pulse (76) is not purely solitonic anymore for equation (73). Indeed, with the substitution \( w = NU \) in (73), where \( N^2 = 1/(1 - \frac{1}{2}b^2) \), we arrive at the standard NLS equation for \( w \),

\[
  i \frac{\partial w}{\partial z} + \frac{1}{2} \frac{\partial^2 w}{\partial t^2} + w |w|^2 = 0.
\]

with the "renormalized" initial condition

\[
  w(t, 0) = \frac{bN}{\cosh(bt)}.
\]

The direct scattering problem of the NLS equation (77), with the initial condition (78), is well analysed. The conditions for soliton creation, as well as the parameters of the created solitons, are known (see, e.g. ref. [23]). As follows from [23], the input pulse (78) creates one soliton of the form,

\[
  U(t, z) = \frac{a \exp(\frac{i}{2}a^2z)}{\cosh(at)},
\]

provided \( \frac{1}{2} < N < \frac{1}{2} \). The soliton amplitude \( a \) is determined by the relation \( a = b(2N - 1) \). Thus, if one starts from the input pulse (76), such a pulse creates an 'average' soliton (79) with the amplitude \( a \) which does not coincide with the amplitude of a soliton which is created by the same input pulse in a primary (unperturbed) NLS model. The resulting shift of the soliton amplitude \( \Delta a \) is

\[
  \Delta a = 2b(N - 1) = 2b \left( \frac{1}{\sqrt{1 - (\frac{\delta}{2})^2}} - 1 \right).
\]

For the small \( \delta \) this result yields \( \Delta a \approx \frac{1}{2}b\delta^2 \).

Analytical predictions presented above have been checked in ref. [19] by numerical simulations. The input pulse has been selected as purely solitonic, \( U(0, t) = b \text{sech}(bt) \), for
different values of $b$, and the amplification function was taken in the simplest form $\epsilon(z) = \epsilon \cos(qz)$ with $q = 2\pi$ with various gain amplitudes $\epsilon$. The results of the numerical simulations are shown in Figs 3 and 4. In particular, Fig. 3 demonstrates the input pulse dynamics for $b = 1$ and $\epsilon = 2.0$. In Fig. 3(a) the solution $u(z, 0)$ of the full NLS equation (58) is shown by the highly oscillatory curve, which is compared with the corresponding solution $U(z, 0)$ of the averaged equation (73) (smooth curve). As may be concluded by such a comparison, the corresponding solution of the renormalized NLS equation (73) accurately describes the evolution of the pulse averaged over fast oscillations. As is clearly seen in Fig. 3 (see also Fig. 4), the mean value (with smooth oscillations) differs from the value $b = 1$; this difference is caused by the renormalization value of the soliton amplitude. The formula (80) yields $\Delta a = 0.052$ which is in a good agreement with Fig. 3(a). The corresponding contour plot shown in Fig. 3(b) clearly indicates that the averaged pulse

![Fig. 3. The evolution of the pulse $u(0, t) = \text{sech} t$ for $q = 2\pi$ and $\epsilon = 2$. Shown are: (a) the function $u(z, 0)$ characterizing the evolution of the pulse amplitude, and (b) a contour plot of the whole solution $u(z, t)$. The rapidly oscillating curve is given by direct numerical simulations of equation (57) whereas the smooth curve follows from the renormalized NLS equation (73).](image)

![Fig. 4. The same as Fig. 3 but for $\epsilon = 4.0$.](image)
evolution does reproduce quite well the stable soliton dynamics resulting from an effective NLS equation.

As has been proven in ref. [19] by direct numerical simulations, an input pulse (76) emits a small amount of radiation giving an asymptotically stable soliton pulse of a higher amplitude which is a fundamental soliton of the renormalized NLS equation. But for larger gain amplitudes, when the renormalized NLS equation (73) breaks down, the typical dynamical effect observed is the pulse splitting into partial pulses, accompanied by a strong emission and pulse splitting [19].

A renormalization of the input soliton pulse in the presence of loss and a rapidly varying gain is an effect which might be easily corrected by a change of the sechtype input pulse. Indeed, analysing the results (76–78), we conclude that the only pulse which will give a stable soliton propagation is \( U(0, t) = 1/N \cosh(t) [N \) being defined above]. Such an input pulse may be called fundamental for the case of a rapidly oscillating gain, but its parameters are determined by the gain amplitude and frequency.

5. SOLITONS ON SPATIALLY PERIODIC POTENTIALS

The method described above for the case of rapidly varying time-dependent perturbations may be also applied to the case of spatially periodic potentials provided the potential is rapidly changing, i.e. the spatial scale of the perturbations is much smaller than a characteristic width of a soliton. To give a concrete example, in the present section we apply the spatial scale separation with asymptotic expansions to the problem recently discussed in connection with the scale competition in nonlinear systems with spatial disorder [24–26] (see also [27]).

We start from the NLS equation with a periodically varying potential (see ref. [26] and references therein)

\[
i \frac{\partial u}{\partial t} + \frac{1}{2} \frac{\partial^2 u}{\partial x^2} + |u|^2 u = \epsilon u \cos(kx). \tag{81}
\]

Let us consider now the potential as rapidly oscillating, i.e. \( k \) is assumed to be large. In order to derive an effective equation for the slowly varying part of the wave field, we look for a solution of (81) in the form,

\[
u = U + A \cos(kx) + B \sin(kx) + C \cos(2kx) + D \sin(2kx) + \ldots, \tag{82}
\]

where the functions \( U, A, B, \) and etc. are assumed to be slowly varying. The mean value of \( U(x, t) \) during an oscillation period is assumed to be \( u \). Our goal is to derive an effective equation for the function \( U \). To this end, we substitute (82) into (81) and equate the coefficients in front of the different harmonics to obtain the system of the coupled equations [cf. the corresponding equations of Section 4 for the case of time-dependent rapidly varying perturbations],

\[
i \frac{\partial U}{\partial t} + \frac{1}{2} \frac{\partial^2 u}{\partial x^2} + |U|^2 U + (U^* A^2 + U A|^2 \ldots) = \frac{1}{2} \epsilon A, \tag{83}
\]

\[
i \frac{\partial A}{\partial t} + \frac{1}{2} \frac{\partial^2 A}{\partial x^2} - k^2 A + 2k \frac{\partial B}{\partial x} + (U^2 A^* + 2|U|^2 A + \ldots) = \epsilon(U + \frac{1}{2} C + \ldots), \tag{84}
\]

\[
i \frac{\partial B}{\partial t} + \frac{1}{2} \frac{\partial^2 B}{\partial x^2} - k^2 B - 2k \frac{\partial A}{\partial x} + (U^2 B^* + 2|U|^2 B + \ldots) = \frac{1}{2} \epsilon D, \tag{85}
\]

\[
i \frac{\partial C}{\partial t} + \frac{1}{2} \frac{\partial^2 C}{\partial x^2} - 4k^2 C + 4k \frac{\partial D}{\partial x} + (U^2 C^* + U^* A^2 + U A|^2 + 2|U|^2 C + \ldots) = \frac{1}{2} \epsilon A, \tag{86}
\]
and so on. One of the main steps of the subsequent analysis is to find the form of asymptotic expansions for the coefficients $A$, $B$, $\ldots$, which allow to solve the infinite system of the coupled equations (84–86). It may be proved that for this case the asymptotic expansions must be taken in the form,

$$ A = \frac{a_1}{k^2} + \frac{a_2}{k^4} + \ldots, \quad B = \frac{b_1}{k^3} + \ldots, \quad C = \frac{c_1}{k^4} + \ldots, \quad D = \frac{d_1}{k^5} + \ldots, \quad (87) $$

and so on. Substituting (87) into (83–86) and equating the terms of the same orders in $k$, we find

$$ a_1 = -\epsilon U, \quad (88) $$

$$ b_1 = -2\frac{\partial a_1}{\partial x}, \quad (89) $$

$$ a_2 = 2i\frac{\partial a_1}{\partial t} - 3\frac{\partial^2 a_1}{\partial x^2} + 2U^2a_1^* + 4|U|^2a_1^*, \quad (90) $$

and so on.

Applying the expansions (87) to equation (83) for the slowly varying component $U$, we find the equation

$$ i\frac{\partial U}{\partial t} + \frac{1}{2}\frac{\partial^2 U}{\partial x^2} + |U|^2 U + \frac{1}{2}\frac{a_1^2}{k^4} U^* + \frac{|a_1|^2}{k^4} U = \frac{1}{2}\epsilon\left(\frac{a_1}{k^2} + \frac{a_2}{k^4}\right), \quad (91) $$

which after using the results (88–90) takes the form

$$ i\frac{\partial U}{\partial t} + \frac{1}{2}\frac{\partial^2 U}{\partial x^2} + |U|^2 U = -\frac{1}{2}\delta^2 U + \frac{\delta^2}{k^2}\left(-i\frac{\partial U}{\partial t} + \frac{3}{2}\frac{\partial^2 U}{\partial x^2} - \frac{3}{2}|U|^2 U\right). \quad (92) $$

Here it is assumed that the parameter $\delta^2 = \epsilon^2/k^2$ may be, in principle, of order of $\mathcal{O}(1)$. As follows from (92), up to the order $k^{-4}$ the averaged dynamics of the solitons on the rapidly varying periodic background is described by the integrable but renormalized NLS equation. This result does explain a series of surprise numerically observed effects mentioned by Scharf and Bishop [26] who observed that NLS solitons, even being perturbed by a strong but rapidly varying periodic perturbations, behave like standard NLS solitons showing, for example, almost elastic pair collisions (see also the paper by R. Scharf [27] in this issue).

6. CONCLUSIONS

In conclusion, we have proposed a rigorous analytical approach to derive an effective (‘averaged’) nonlinear equation describing the nonlinear dynamics and soliton propagation in the presence of rapidly varying periodic perturbations of different nature. We have demonstrated that in the first order of the asymptotic procedure the effective equation for the slowly varying field component looks like a renormalized nonlinear equation to the primary model. For example, in the case of the direct driving force or rotating (phase-locked) background this is a renormalized SG equation, and it is a double SG equation for the parametric driving force. However, the method itself does allow us to find in a rigorous way the effective equation for the slowly varying field component in any order of the asymptotic expansions. The similar approach is shown to be effective for rapidly varying periodic potentials.

The method we have proposed, as well as our main conclusion that the averaged nonlinear dynamics may be drastically modified by rapidly varying (direct or parametric
driving forces or potentials varying in space), are rather general to be applied to many other nonlinear models, in particular those supporting solitons.

Acknowledgements—Most of the research for this paper has been carried out during the stay of one of the authors (Yu.K.) in Germany as an AvH fellow. Yuri Kivshar acknowledges the financial support of the Alexander-von-Humboldt-Stiftung and a warm scientific atmosphere at Institut für Theoretische Physik I (Universität Düsseldorf). He is also indebted to N. Grønbech-Jensen, R. H. Parmentier, M. L. Quiroga-Teixeiro, F. Rödelisperger, M. Salerno, M. R. Samuelsen, A. Sánchez, and S. K. Turitsyn for useful collaborations, and to R. Grauer and R. Scharf for valuable discussions. Part of this work is supported by the European Community.

REFERENCES