Generalization of Berry’s geometric phase, equivalence of the Hamiltonian nature, quantizability and strong stability of linear oscillatory systems, and conservation of adiabatic invariants

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A linear set of ordinary differential equations with a matrix depending on a set of adiabatically varying parameters is considered. Its asymptotic solutions are constructed to an arbitrary accuracy in the adiabaticity parameter $\varepsilon$. By extending the phase space of the system not only with the space of parameters, like in the theory of Berry’s phase, but also with the space of their derivatives, it proves possible to represent the phase of the solution as a sum of the dynamic phase and a generalized geometric phase (determined by a contour integral in the space of parameters and derivatives). Hence, it is possible to obtain the asymptotic results to any degree of accuracy in $\varepsilon$, while earlier they were obtainable only in the first-order adiabatic approximation. Namely, for linear oscillatory adiabatic systems, the quantizability, parametric stability, and the Hamiltonian nature of the system are equivalent properties. As a consequence, one can obtain the well-known result regarding conservation with accuracy to any power of $\varepsilon$ of adiabatic invariants in a Hamiltonian system on a torus. An important point is that the generalized geometric phase can appear, in contrast to Berry’s phase, with only one real varying parameter. © 2002 American Institute of Physics.

I. INTRODUCTION

This paper develops the ideas of Refs. 1 and 2 that proposed to analyze nonautonomous dynamic systems with the help of a functional approach. In this case the phase space of a dynamic system is extended not only by adding the dimension of time but also a space of parameters varying with time. As a result, solutions to the equations, changes of variables, etc., are considered as functionals (generally speaking, nonlocal) of the parameters and not just as functions of time. As it occurs, the approach allows one to obtain essential results of considerable interest. For example, the theory of Berry’s phases provides a prominent example of the functional approach to nonautonomous dynamic systems.

Note that an essential feature of the functional approach is the possibility to separate coordinate substitutions into local and nonlocal ones. The local substitutions involve only current values of system parameters and their derivatives, and hence cannot result in qualitative distortions of the system’s phase portrait. Whereas nonlocal substitutions are, for the most part, integral functionals of parameters containing the system’s prior history. Such substitutions always distort the global qualitative phase portrait.

Reference 1 was an attempt to develop a consistent theory of linear adiabatic systems from the viewpoint of the functional approach. Special attention was paid to oscillatory systems, i.e., the systems with purely imaginary current eigenvalues. For such systems, adiabatic equivalence was proven (i.e., equivalence to the first order in the adiabatic parameter $\varepsilon$) of the quantizability, strong stability, and Hamiltonian property of the system. It was exactly the functional approach that made
the proof possible, and the notion of complex geometric phases was used essentially.

In the present paper we will show that a consistent application of the technique permits extending the formalism up to any order of \( \varepsilon \). As a result, the notion of the *generalized geometric phase* is introduced, and the above-mentioned statements of Ref. 1 are proved in a similar way, however to a new accuracy. It is important that the generalized geometric phase, while being described analogously to Berry’s phase and representing similar physical effects, can appear under adiabatic changes of only one real parameter. In this paper, some modifications of the strong stability property are introduced for parameter-dependent systems in a more rigorous way than in Ref. 1, namely the parametric stability and strong stability in a class. It is noteworthy that from the formalism introduced and the theorems proved in the paper some well-known results follow immediately. Namely, adiabatic invariants of the Hamiltonian system whose phase trajectories lie, under constant parameters, on an \( n \)-dimensional torus, remain unchanged with accuracy to any power of \( \varepsilon \).4–6

The central idea of this work, allowing an efficient extension of the known results to any degree of accuracy, consists in analyzing the system in a generalized space of parameters that involves not only the dimensions of the parameters but of their derivatives as well.

**II. GENERAL FORMALISM**

Consider the linear system described by an \( n \)-dimensional vector equation in \( \mathbb{C}^n \),

\[
x' = A(\vec{\mu})x.
\]

[Let us note that all conclusions of this paper can be obtained in a similar way for the initial system in \( \mathbb{R}^{2n} \). Besides, the results can be derived in an almost identical way for the initial system with the matrix of form \( A = \mathbf{A}(\vec{\mu}, \vec{\mu}', \ldots, \vec{\mu}^{(m)}) \).] Here \( x = (x_1, \ldots, x_n) \) is the vector-function to be found (its components will be called coordinates), the prime stands for differentiation with respect to the independent real variable \( t \) (further referred to as time), \( A(\vec{\mu}) \) is a linear operator defined by a square nondegenerate matrix, which is an infinitely differentiable function of the set of \( s \) adiabatically varying parameters \( \vec{\mu} = (\mu_1, \ldots, \mu_s) \), and \( \vec{\mu} \) is a vector from a compact simply connected domain \( \varphi \subset \mathbb{R}^s \).

The eigenvalues of the matrix \( A(\vec{\mu}) \) are assumed to satisfy the following conditions for all values of the parameters:

\[
0 < |\lambda_j(\vec{\mu})| < \infty, \quad j = 1, \ldots, n; \quad \lambda_j(\vec{\mu}) \neq \lambda_l(\vec{\mu}) \quad \text{for} \quad j \neq l \tag{1a}
\]

The latter requirement is necessary to permit the use of independent adiabatic solutions of (1). [Applicability criteria of the adiabatic approximation for the case when eigenvalues are closing in can be found in Ref. 7.]

The system parameters are subject to an arbitrary adiabatic variation with time:

\[
\vec{\mu} = \bar{\mu}(\varepsilon t), \quad \varepsilon \ll 1.
\]

Here, \( \varepsilon \) is the adiabaticity parameter, and the vector-function \( \bar{\mu}(\varepsilon t) \) is assumed to be infinitely differentiable. From here on, referring to system (1), we also imply all the above-listed requirements.

Notice that there is no need to require the nonresonance character of the \( \bar{\mu}(\varepsilon t) \) dependence as was done in Ref. 1. The point is that even in the presence of parametric resonance in the adiabatic oscillatory system the growth rate is exponentially small,8,9 being a value of the form \( \alpha \exp(-\beta \varepsilon^{-1}) \). [In the case of parametric resonance with nonadiabatic sections of the function \( \bar{\mu}(t) \), the growth rate generally is of order \( \varepsilon^1 \), where \( \varepsilon \) is the reciprocal of the period.7] For this reason, parametric resonances are beyond the scope of this paper treating power-series approximations in \( \varepsilon \). Furthermore, the statements and theorems discussed here utilize the function \( \bar{\mu}(\varepsilon t) \) only as an auxiliary tool. In fact they deal with the system as a whole, i.e., structure of the \( A(\vec{\mu}) \) dependence.
To construct asymptotic solutions to system (1) we will use the following idea proposed by Neishtadt.\textsuperscript{10} (Compare also with the method of successive diagonalizations in Ref. 4.) Let us bring the parameter-dependent matrix $A(\vec{\mu})$ to a diagonal form by applying a linear substitution of coordinates. In terms of new coordinates, the matrix of the transformed system will be only slightly nondiagonal (the off-diagonal terms are proportional to the first-order time derivatives of parameters and have the order of $\epsilon$). Let us further diagonalize the matrix obtained to a first order in $\epsilon$ through the next linear substitution of coordinates. In these coordinates the off-diagonal matrix elements will be small terms of order $\epsilon^2$ being proportional to second-order time derivatives. This procedure can be similarly reiterated. Thus, the initial system can be reduced, for any finite order in $\epsilon$, to a diagonal form, for which exact solutions can be easily written. The sequence of recurrence relations to describe the procedure is given in Appendix A.

Equation (1), upon $k$ successive diagonalizations, takes the form

$$x_{[k]}'=[A_{[k]}+\Delta_{[k]}]x_{[k]},$$

where the variables $x_{[k]}$ are related to the initial ones through a certain reversible local substitution without singularities of the form $x=B(\vec{\mu},\vec{\mu}',...,\vec{\mu}^{(k)})x_{[k]}$; the subscript in the square brackets stands for the diagonalization number; $A_{[k]}$ is the diagonal matrix containing terms of order $\epsilon^k$ and under, while the components of the matrix $\Delta_{[k]}$ are of order $\epsilon^{k+1}$. The matrix $A_{[k]}$ can be written in the form [see (A5)]

$$A_{[k]}=\Lambda+\sum_{j=0}^{k-1}dg\Delta_{[j]}.$$ (4)

Here $|\Delta_{[j]}|\sim\epsilon^{j+1}$, the operator $dg$ rejects off-diagonal components of the matrix, and $\Lambda=A_{[0]}=\text{diag}(\lambda_1,...,\lambda_n)$, with the values $\lambda_j=\lambda_j(\vec{\mu})-\epsilon^0$ being current eigenvalues of the matrix $A(\vec{\mu})$ of the initial equation (1).

Let us determine the form of the matrices $\Delta_{[j]}$. In view of their proportionality to the corresponding power of $\epsilon$ and with account of Eq. (2), it can be seen that $\Delta_{[1]}$ is proportional to the first-order time derivative of $\vec{\mu}(\epsilon t)$ and has the general form like

$$\Delta_{[0]}=\bar{F}_{[0]}(\vec{\mu}(\epsilon t))\vec{\mu}'(\epsilon t),$$

where $\bar{F}_{[0]}(\vec{\mu}(\epsilon t))$ is the matrix field over the $\vec{\mu}$-space of parameters. The nonpotential component of this field gives rise to Berry’s geometric phase, or Hannay’s angle (see Refs. 1 and 3). Similarly, $\Delta_{[2]}$ can be written in general form as

$$\Delta_{[1]}=\bar{F}_{[1]}(\vec{\mu}(\epsilon t))\vec{\mu}'(\epsilon t)+\bar{F}_{[1]}(\vec{\mu}(\epsilon t))(\vec{\mu}'(\epsilon t))^2,$$

where $\bar{F}_{[1]}(\vec{\mu}(\epsilon t))$ also is a matrix field and $\bar{F}_{[1]}(\vec{\mu}(\epsilon t))$ is a scalar (with respect to the $\vec{\mu}$-space) matrix-function.

Continuing the procedure, we see that the diagonal matrix $dg\Delta_{[j]}$ can be represented as a sum of elements that are proportional to different orders of derivatives of $\vec{\mu}(\epsilon t)$. Besides, in each term, the sum of products of the derivative order by the power it is raised to is equal to $j+1$.

Now we turn to solving Eq. (3). Its solutions can be written as follows:

$$x_{[k]}(t)\approx x_{[k]}(0)\exp\left[\int_0^t \left[\Lambda+\sum_{j=0}^{k-1}dg\Delta_{[j]}+O(\epsilon^{k+1})\right]d\tau\right].$$ (5)

From what has been said previously on the form of the matrices in the integrand, it appears that the solution (5) can be written like
sents a field of diagonal matrices over this space. Finally, similarly to the geometric phase theory, the integration contour in the $M$ is determined only by boundary values of the parameters and their derivatives, and is independent of the solutions, where $\bar{M}$ is the contour along which the representative point of the system moves in the generalized parameter space.

The solution $x_{i[k]}(t) = x_{i[k]}(0) \exp \left[ \int_0^t \left( \lambda_i(\bar{\mu}(\epsilon \tau))d\tau + \int_L \bar{F}(\bar{M}(\epsilon \tau))d\bar{M} \right) + O(\epsilon^{k+1}t) \right]$, (7)

where $L$ is the contour along which the representative point of the system moves in the generalized parameter space.

The solution (7) is fully identical in its form to the known adiabatic solution [Eq. (8) of Ref. 1] that was specified to within $\epsilon^1$, however its accuracy is $\epsilon^k$. The first term in Eq. (7) is the standard “fast” dynamic phase. The second term is the “slow” phase, whose nonlocal part represents the generalized geometric phase. It has the form of the standard Berry phase, however written in the generalized parameter space.

Equation (7) can be easily separated by the degrees of freedom of the system in $\mathbb{C}^n$, because it contains diagonal matrices alone. From here on, we will separately consider the projections of the solutions, $x_{i[j]}(t) = x_{i[j]}(0) \exp \left[ \int_0^t \lambda_j(\bar{\mu}(\epsilon \tau))d\tau + \int_L \bar{f}_j(\bar{M})d\bar{M} \right] + O(\epsilon^{k+1}t)$, (8)

where $j = 1, \ldots, n$ and $\bar{f}_j(\bar{M}) = \mathbf{F}_j(\bar{M})$.

Let us isolate the potential component of $\bar{f}_j(\bar{M})$, representing the field as
\[
\bar{f}_j(\bar{M}) = \text{grad}[\varphi_j(\bar{M})] + \bar{f}_j^{c}(\bar{M}),
\]

where $\varphi_j(\bar{M})$ is a scalar function, while $\bar{f}_j^{c}(\bar{M})$ is the nonpotential component of $\bar{f}_j(\bar{M})$. By substituting Eq. (9) into Eq. (8), we obtain
\[
x_{i[j]}(t) = x_{i[j]}(0) \exp \left[ \int_0^t \lambda_j(\bar{\mu}(\epsilon \tau))d\tau + \left( \varphi_j(\bar{M}(\epsilon \tau)) - \varphi_j(\bar{M}(0)) \right) + \int_L \bar{f}_j^{c}(\bar{M})d\bar{M} \right] + O(\epsilon^{k+1}t).
\]

The contribution of the potential field component $\bar{f}_j(\bar{M})$ to the solutions (8), (10) is local. It is determined only by boundary values of the parameters and their derivatives, and is independent of the integration contour in the $\bar{M}$-space. Therefore, it is the component that determines the connection between the slow amplitude of the solution and current values of parameters and their derivatives, and is responsible for the construction of adiabatic invariants.

The contribution of the nonpotential component, $\bar{f}_j^{c}(\bar{M})$, is, on the contrary, nonlocal. It depends essentially on the geometry of the contour $L$ (i.e., on all prior values of the parameters), being nonzero even for cyclic variations of the parameters that correspond to closed contours in the $M$-space. The last term in the exponent of Eq. (10) is the complex generalized geometric phase.

For the convenience of further consideration we will separate the real and the imaginary part of the generalized geometric phase. Its imaginary part will be called the generalized geometric phase proper,
$$\psi_j(L) = \text{Im} \int_L \tilde{f}_j^{(c)}(\tilde{M}) \mathrm{d}\tilde{M},$$  

while its real part will be spoken of as the generalized geometric amplitude,

$$\gamma_j(L) = \text{Re} \int_L \tilde{f}_j^{(c)}(\tilde{M}) \mathrm{d}\tilde{M},$$

since it is exactly its changes that bring forth the variations of the solution amplitude. Note that the \(M\)-space is multidimensional even in the case of only one varying real parameter in the system \((s = 1)\). For this reason generalized geometric phases and amplitudes may appear even when Berry’s phase certainly does not exist (the \(\bar{M}\)-space is one-dimensional).

The potential component of \(\tilde{f}_j(\tilde{M})\) can be easily reduced to zero via the local substitution \(x_{[k\lambda]}(t) = \exp[\varphi_j(\bar{M}(\varepsilon t)) - \varphi_j(\bar{M}(0))]z_j(t)\). As a result, from (10) we obtain:

$$z_j(t) = z_j(0) \exp \left\{ \int_0^t \lambda_j(\bar{M}(\varepsilon \tau)) \mathrm{d}\tau + \int_L \tilde{f}_j^{(c)}(\tilde{M}) \mathrm{d}\tilde{M} \right\} + O(e^{k+1}t).$$  

We will also write the equation corresponding to projections of Eq. (3) in terms of these variables:

$$z_j' = (\lambda_j + \tilde{f}_j^{(c)}(\bar{M}'))z_j + O(e^{k+1}t).$$

These expressions coincide in form with those obtained in Ref. 1 [Eqs. (7)–(14) of this work correspond to Eqs. (8), (11), (13)–(16), (20), and (19) of paper Ref. 1]. In what follows we will derive analogous results, therefore in the following discussion we will only write out the fundamental propositions, referring the reader to Ref. 1 for details of the proofs. All one needs to do is to replace the parameter space and geometric terms with their generalized analogs.

### III. MAIN RESULTS

#### A. Global qualitative portrait of the system

**Statement I:** The generalized geometric terms, together with the current eigenvalues \(\lambda_j\), determine the qualitative portrait of a linear adiabatic system.

Indeed, to describe the system we can introduce, similar to Ref. 1, the efficient eigenvalues corresponding to the solutions (13) over a given time interval,

$$\lambda_j^{\text{eff}}(t) = \frac{\ln z_j(t) - \ln z_j(0)}{t}.$$  

By substituting the solutions (13) into Eq. (15), we obtain:

$$\lambda_j^{\text{eff}}(t) = (\lambda_j)_j + \frac{\gamma_j + i\psi_j}{t} + O(t^{-1}, e^{k+1}),$$

where \((\lambda_j)_j = (1/t) \int_0^t \lambda_j(\bar{M}(\varepsilon \tau)) \mathrm{d}\tau\) are average values of the current eigenvalues. The second term in Eq. (16) contains terms of different orders ranging from \(e\) to \(e^k\) and, generally speaking, do not tend to zero together with \(t^{-1}\), since during continuous circulation of the system’s representative point in the \(\bar{M}\)-space the geometric terms increase infinitely. If there is a limit \(\lambda_j^{\text{eff}} \equiv \lim_{t \to \infty} \lambda_j^{\text{eff}}(t)\), then the behavior of the solutions for large times (strictly speaking, for times large enough so that \(\lambda_j^{\text{eff}}(t)\) does not leave the small vicinity of its limiting value) can be “roughly” (i.e., without account of local deviations) described by the formula \(z_j(t) = z_j(0) \exp(\lambda_j^{\text{eff}}(t))\). For example, under periodic (of period \(T \sim e^{-1}\)) variations of the parameters the value \(\lambda_j^{\text{eff}}(t)\) tends asymptotically to the limit...
where $L_T$ is the closed contour corresponding to one period. The above-derived solutions as well as the analysis presented are applicable over times $t \leqslant e^{-(k+1)}$. It is significant that the system’s portrait analyzed in $z$-coordinates is qualitatively equivalent to the portrait of the initial system (1). Indeed, we used only local coordinate substitutions that leave the system’s fundamental characteristics (current eigenvalues and generalized geometric phases, i.e., efficient eigenvalues) invariant (see Ref. 2 and Appendix B in Ref. 1).

B. Adiabatic and integral invariants

Statement II: A degree of freedom of the system possesses an adiabatic invariant of action, conserved to the order of $\varepsilon^k$, if and only if

$$\text{Re} \lambda_j = 0,$$

(17)

$$\gamma_j = 0.$$

Indeed, if the conditions (17) are met, then Eq. (13) yields

$$I_j(t) = |z_j(t)|^2 \approx \text{const.}$$

(18)

The values $I_j$ are known as the adiabatic invariants of action. Each $I_j$ corresponds to conservation of the phase flux of the $j$th degree of freedom in the $z$-space. In terms of the initial coordinates, the adiabatic invariants determine, under given initial conditions, an unambiguous relation between the amplitude of the approximate solution and current values of the parameters. Note that the conditions (17) are equivalent to the requirement that the real part of the effective eigenvalue $\lambda_j^{\text{eff}}$ specified previously, be equal to zero. In other words, this is the requirement of conservation of oscillatory nature of the system movement over large times with adiabatically varying parameters.

Following Ref. 1, it can be easily shown that the existence of Poincaré’s first integral invariant is equivalent to the presence in the system of a complete set of $n$ adiabatic invariants (once again, this is true up to the order of $\varepsilon^k$). [It should be noted at this point that we do not consider the case of Hamiltonian systems with $\text{Re} \lambda_j \neq 0$. Such systems also possess Poincaré’s integral invariants, however adiabatic invariants are absent. The complex structure suggested here and integral invariants in these systems have different form as compared with Hamiltonian systems on a torus ($\text{Re} \lambda_j = 0$), which mostly are considered in this paper.] In addition, the condition for conservation of the system’s phase volume to the accuracy $\varepsilon^k$ (Poincaré’s $n$th integral invariant, or Liouville’s theorem) are:

$$\sum_{j=1}^{n} \text{Re} \lambda_j = 0,$$

(19)

$$\sum_{j=1}^{n} \gamma_j = 0.$$

C. Hamiltonian formalism

We will further consider oscillatory systems, i.e., such systems that

$$\text{Re} \lambda_j = 0, \quad j = 1,...,n.$$
Theorem I: The system (1) with purely imaginary current eigenvalues [Eq. (20)] coincides, to accuracy \(\varepsilon^k\), with a Hamiltonian system if and only if all of its generalized geometric amplitudes are equal to zero:

\[
\gamma_j = 0, \quad j = 1, \ldots, n.
\]  

(21)

The Hamiltonian of such a system is represented by the Hamiltonian of a sum of independent oscillators:

\[
H(p, q) = \sum_{j=1}^{n} H_j(p_j, q_j) = \frac{1}{2} \sum_{j=1}^{n} \omega_j (p_j^2 + q_j^2),
\]

(22)

where \(i \omega_j = \lambda_j + \tilde{f}_j(c) \tilde{M}\).

Carrying out the realification of z-space, \(p_j = \text{Re} z_j\) and \(q_j = \text{Im} z_j\), we obtain from Eq. (14) with account of Eqs. (20) and (21),

\[
p_j' = -\text{Im}(\lambda_j + \tilde{f}_j(c) \tilde{M})q_j + O(\varepsilon^{k+1}),
\]

\[
q_j' = \text{Im}(\lambda_j + \tilde{f}_j(c) \tilde{M})p_j + O(\varepsilon^{k+1}).
\]

(23)

To within the order \(\varepsilon^k\), this is a Hamiltonian system with the Hamiltonian (22). Thus, the sufficiency of the condition (21) for the system to be a Hamiltonian one has been proven. The necessity of condition (21) can be proved following Ref. 1 by replacing the parameter space and the geometric values involved with their generalized analogs (also see Appendix B).

Note that points of the unperturbed system trajectory (in the proofs of Theorems I and III) should not belong to the boundary of the considered compact region \(\nu\) in the \(\tilde{\mu}\)-space.

D. System quantization

Theorem II: System (1) with purely imaginary current eigenvalues [Eq. (20)] coincides, to accuracy \(\varepsilon^k\), with a quantizable system if and only if all of its generalized geometric amplitudes are equal to zero [Eq. (21)].

The proof can be performed following Ref. 1 by replacing the parameter space and the geometric values involved with their generalized analogs.

E. Strong stability properties

Now we turn to stability properties of the systems considered. In Ref. 1 equivalence of the Hamiltonian nature and quantizability of the system to a special version of strong stability was proven. Here we will introduce the property of a parametric stability, which we think is more adequate when discussing parametric systems. Next, we will formulate a strong stability property, which could be more properly referred to as strong stability in a class. It will be shown that the parametric stability is a primary property for systems with parameters. The strong stability property can be introduced only for the class of parametrically stable systems.

Definition I: System (1) with a specific (rather than arbitrary) \(\tilde{\mu}(\varepsilon t)\) dependence is said to be a realization of system (1).

In essence, the notion of realization emphasizes that in the functional approach we operate in fact with a set of systems characterized by a variety of \(\tilde{\mu}(\varepsilon t)\) dependencies, whereas in the temporal approach we consider a single system.

Definition II: Let a realization \(\tilde{\mu} = \tilde{\mu}_0(\varepsilon t)\) of system (1) with the matrix \(\mathbf{A}(\tilde{\mu})\) be stable, i.e., all of its solutions are bounded, if there exists a \(\delta > 0\), such that any perturbed realization \(\tilde{\mu} = \tilde{\mu}_1(\varepsilon t)\) of the same system satisfying the inequality

\[
|\tilde{\mu}_1(\varepsilon t) - \tilde{\mu}_0(\varepsilon t)| < \delta, \quad \forall t,
\]

(24)
is also stable, then the initial realization of the system is called parametrically stable.

The system is parametrically stable if all of its realizations are parametrically stable.

As noted at the beginning of Sec. II, parametric resonances in adiabatic systems go beyond the accuracy of the approximation considered here. Talking of stability here and in the following, we will not consider parametric resonances again, since we operate in the framework of the approximation adopted.

**Definition III:** Let system (1) with the matrix \( A = A_0(\bar{\mu}) \) from a certain class \( \mathcal{N} \) be parametrically stable. If there exists a \( \delta > 0 \), such that any perturbed system (1) with the matrix \( A = A_1(\bar{\mu}) \) from the same class \( \mathcal{N} \), satisfying the inequality

\[
\| A_1(\bar{\mu}) - A_0(\bar{\mu}) \| < \delta, \quad \forall \bar{\mu} \subset \mathcal{N},
\]

is also parametrically stable, then the initial system is called strongly stable in the class \( \mathcal{N} \).

In essence, the property of parametric stability reflects the stability of a system against small variations of its parameters. The property of strong stability in a class describes stability under small variations in the system matrix, however, demanding that the matrix does not leave the given class, which is to be determined in every particular problem. Like in Ref. 1, we speak here of strong stability in the class of Hamiltonian oscillatory systems with different eigenvalues. (This kind of stability was proven previously for systems with periodic coefficients.\(^{12-13}\)) Apparently, small variations in the system matrix that take the system out of that class will readily produce loss of the system’s stability. This is the reflection of the fact that Hamiltonian oscillatory systems are structurally unstable.\(^{13}\) More specifically, when the system loses its oscillatory nature, it may become simply unstable (parametrically unstable even more so) even for fixed values of its parameters (if Re \( \lambda_j > 0 \)). At the same time, if a system leaves the class of Hamiltonian ones, while retaining its oscillatory nature, then the system becomes parametrically unstable (a property opposite to the one of parametric stability).

**Remark:** If we choose a class in which all the systems are parametrically stable, then they will be also strongly stable in the class.

As a consequence of the Remark, it is the parametric stability property that can be naturally assumed to be primary: it determines the class of systems in which they possess the property of strong stability.

**Theorem III:** System (1) with purely imaginary current eigenvalues [Eq. (20)] is parametrically stable to the order of \( \varepsilon^k \) if and only if its generalized geometric amplitudes are all equal to zero [Eq. (21)].

Once again, the proof is carried out following the pattern of Theorem III in Ref. 1, however replacing the parameter space and the geometric values involved with their generalized analogs (also see Appendix B).

**F. Corollaries**

By virtue of Eq. (21) representing the necessary and sufficient condition for all of the above-given theorems, the following conclusions can be drawn.

**Corollary I:** The Hamiltonian nature, parametric stability, and quantizability are equivalent properties with accuracy to any power of \( \varepsilon \), for system (1) with purely imaginary current eigenvalues.

Also, the property of parametric stability (Theorem III) determines the strong stability class of the oscillatory systems (1):

**Corollary II:** Systems (1) are strongly stable, with accuracy to any power of \( \varepsilon \), in the class of oscillatory Hamiltonian systems. Moreover, the class of Hamiltonian systems is the complete strong stability class among the oscillatory systems (1), i.e., none of the oscillatory systems not belonging to this set is strongly stable.
It is clear that the complete class of strongly stable systems (1) represents a union of the set of Hamiltonian systems with $\text{Re}\lambda_j=0, j=1,...,n$, and the set of systems with a negative real part of current eigenvalues $(\text{Re}\lambda_j<0, j=1,...,n)$. The strong stability in the latter set is clear enough and is not discussed here.

Corollary III: Let the phase curves of a linear Hamiltonian system for fixed values of parameters represent the winding of an $n$-dimensional torus with various rotation periods. Then $n$ adiabatic invariants exist in the system that are retained during slow variations of the system parameters with accuracy to any power of $\varepsilon$.

Indeed, if the system is a Hamiltonian one and the conditions (20) are met, then its generalized geometric amplitudes are equal to zero (Theorem I) for any $k$. And hence (Statement II), the system possesses a set of $n$ adiabatic invariants that are conserved to within the accuracy $e^k$ with arbitrary $k$.

IV. CONCLUSIONS

The paper considered a linear dynamic system of the general kind, described by the complex vector equation (1) with a nondegenerate matrix possessing different nonzero eigenvalues. The system matrix is assumed to depend on a set of parameters varying adiabatically in time. The principal results reported in the paper are as follows:

1. A procedure has been suggested for constructing asymptotic solutions (5) that satisfy the system (1) to terms of any given order in $\varepsilon$.

2. The notion of the generalized parameter space $\tilde{M}=(\tilde{\mu}(et),\tilde{\mu}'(et),...,\tilde{\mu}^{(k-1)}(et))$ has been introduced that allows representing asymptotic solutions as an exponential function of the sum of the dynamic phase and the generalized geometric phase. The latter can be represented as a contour integral of some field on the $M$-space. Thus the problem has been formally reduced to the case (already considered in Ref. 1) of adiabatic solutions determined with an accuracy of $\varepsilon$.

The $M$-space is multidimensional even in the case of a single real varying parameter in the system ($s=1$). Hence, generalized geometric phases can appear when Berry’s phase is certainly absent. The appearance of such phases may produce interesting effects in physical systems that are similar to the appearance of Berry’s phase for higher orders of $\varepsilon$. [In the papers now being prepared for publication the author shows that the generalized geometric phase can appear even in very simple systems, such as an oscillator with a varying eigenfrequency or the rotating polarization plane of light in a one-dimensionally inhomogeneous medium.]

3. Making use of the analogy with the adiabatic problem already analyzed to within $\varepsilon$, we have shown the generalized geometric terms, along with current eigenvalues, to be fundamental characteristics of the system. They determine its global qualitative portrait over times up to $e^{-k}$. The real part of the complex generalized geometric phase, i.e., the generalized geometric amplitude, plays a special role in the system analysis. The geometric amplitudes determine the system’s Lyapunov indices.

4. The necessary and sufficient conditions (17) for constructing the adiabatic invariant for one degree of freedom have been formulated (under the assumption that the invariant is conserved to within $e^k$). These are the absence of the generalized geometric amplitude and zero value for the real part of the current eigenvalue for the given degree of freedom. Within the same approximation, Poincaré’s first integral invariant is equivalent to the complete set of $n$ adiabatic invariants. In other words, the conditions for its existence are an oscillatory nature of the system and zero values for all geometric amplitudes. The conditions of the Liouville theorem (the $n$th Poincaré integral invariant) demand that real parts of the current eigenvalues be zeros, as well as the sum of generalized geometric amplitudes.

5. The central results of the paper are the following equivalence relationships. For the oscillatory systems (1) equivalence has been proved of the Hamiltonian nature of the system, its quantizability, and parametric stability with accuracy to any power of $\varepsilon$. These properties are implemented if and only if the generalized geometric amplitudes of the oscillatory system are all equal to zero.
Particularly discussed have been the properties of parametric stability and strong stability in a class that are modifications of the strong stability property as applied to parametric systems. As has been shown, it is natural to assume the parametric stability as the primary property. It determines the class of strong stability of parametric systems. In the case under consideration, the complete class of strongly stable oscillatory systems is formed by Hamiltonian oscillatory systems.

It is of interest that a direct consequence of the theorems proven here is the known result concerning adiabatic invariants. Namely, a Hamiltonian system with constant-parameter phase trajectories lying on an \( n \)-dimensional torus supports a complete set of \( n \) adiabatic invariants that are conserved with accuracy to any power of \( < \!4^{-6} \).

It appears to us that the equivalence relationships proven here and in Ref. 1 do not just have an abstract, mathematical meaning but also possess a physical if not philosophical significance. They show the role of the Hamiltonian nature of our world. It is well known that the principal, elemental oscillatory systems are described by Hamiltonian equations equally in quantum and classical mechanics and in the wave theory. One could think that the form of these equations is somewhat accidental and they might well have different form. However, the results presented in this work offer a clear view of the following fact: if the equations for linear physical oscillators had not been of Hamiltonian nature, then our world would not have been so stable against small perturbations. Besides, it would not have been of quantum nature either, or at least there would have not existed such a correlation between the micro- and macroworld theories.

In conclusion, we stress once again that the derivation of all of these results has become possible exclusively due to the functional approach to nonautonomous systems see Sec. I and Refs. 1 and 2.

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APPENDIX A: CONSTRUCTION OF ASYMPTOTIC SOLUTIONS

Consider in more detail Neishtadt’s method of successive diagonalizations proposed in Sec. II (Compare also with method in Ref. 4.). Let upon the \( k \)th diagonalization the initial equation (1) take the form

\[
x'_{[k]} = [\Lambda_{[k]} + \Delta_{[k]}]x_{[k]},
\]

(A1)

where \( |\Delta_{[k]}| \sim e^{k+1} \) and \( \Lambda_{[k]} \) is a diagonal matrix containing the terms up to the order \( e^k \) inclusive. Next let us perform the \((k+1)\)th diagonalization. Let us make the substitution

\[
x_{[k]} = D_{[k]}x_{[k+1]},
\]

(A2)

where

\[
D_{[k]ij} = \frac{\Delta_{[k]ij}}{(\Lambda_{[k]jj} - \Lambda_{[k]ii})}, \quad j \neq l; \quad D_{[k]jj} = 1.
\]

(A3)

The substitution (A2) always exists and allows inversion, because it is close to the unit matrix (the diagonal elements of \( \Lambda_{[k]} \) are different according to the initial assumptions).

After the substitution (A2), Eq. (A1) will take the form

\[
x_{[k+1]} = D_{[k]}^{-1}[\Lambda_{[k]} + \Delta_{[k]}]D_{[k]}x_{[k+1]}. \]

(A4)

Then we set in (A4)

\[
\Lambda_{[k+1]} = \Lambda_{[k]} + \Delta_{[k]}, \quad (A5)
\]
By substituting Eq. (A3) into Eq. (A6) we can easily verify that the components of $D_{\{k\}}^{\prime+1}$ are proportional to the derivatives of the $D_{\{k\}}$ components, i.e., they are terms of order $\varepsilon^{k+2}$. Finally, by substituting Eqs. (A5) and (A6) into Eq. (A4) we obtain:

$$x_{k+1}^{\prime} = [A_{k+1} + \Delta_{k+1}^{\prime}]x_{k+1}.$$  \hspace{1cm} (A7)

Thus, we have completely described the recurrent step from Eq. (A1) to Eq. (A7). It only remains to derive the initial equation of the form (A1) to apply the induction.

Since the matrix $A$ of the initial equation (1) possesses different nonzero eigenvalues, there exists a matrix $D$ such that $D^{-1}AD = \Lambda$ (The matrix $\Lambda$ was introduced in Sec. II.) By carrying out the substitution $x = Dx_{[0]}$, we obtain from Eq. (1)

$$x_{[0]}^{\prime} = [\Lambda - D^{-1}D']x_{[0]}.$$  \hspace{1cm} (A8)

By setting $A_{[0]} = \Lambda$ and $\Delta_{[0]} = -D^{-1}D'$, we obtain Eq. (A1) for $k=0$. Equations (A2)–(A7) allow one to repeat the diagonalization procedure as many times as necessary, reducing the initial equation to a diagonal form with accuracy to any power of $\varepsilon$.

**APPENDIX B: ON PROPERTIES OF THE $\tilde{M}$-SPACE AND SYSTEM TRAJECTORIES IN IT**

It should be noted that the dimensions of the generalized parameter space ($\tilde{M}$-space), strictly speaking, are not independent. Indeed, consider, for instance, a closed oriented contour of the system’s representative point in the plane $(\mu_j, \mu_j')$ (Fig. 1). It is easy to see that the contour shown by a dashed line cannot be realized. Along the sections of decreasing $\mu_j$, the value $\mu_j'$ should be negative, and hence this section of the contour should lie in the lower half-plane. The contour depicted in Fig. 1 by a solid line meets the requirement, and thus it can be realized.

By generalizing the example, we can suggest the following requirement, which does not, however, exhaust the relations between the coordinates in the $\tilde{M}$-space (certain constraints are also imposed on the trajectories in the planes of $(\mu_j^{(i)}, \mu_j^{(i-2)})$ type and others). At the same time, only the relations considered in the following are essential for proving the theorems of this paper.

Statement: Let $\tilde{M} = \tilde{M}(\tau)$ be some parametrization of a trajectory of the system’s representative point in the $\tilde{M}$-space. Also, let the $\tau$ parameter be a smooth, monotonically increasing function of the real time $t$ ($d\tau/dt > 0$). Then the following equality should hold at any point of the trajectory:

$$\mu_j^{\prime} = \mu_j.$$  \hspace{1cm} (A9)

**FIG. 1.** Realizable (solid line) and nonrealizable (dashed line) contours of the system’s representative point in a $(\mu_j, \mu_j')$ plane of the $\tilde{M}$-space.
for all components $j = 1,...,n$ and all the derivatives $l = 1,...,k-1$ used.

Let us analyze how the constraints (B1) imposed on the coordinates of the $\bar{M}$-space affect the proofs of Theorems I–III. To this end, first note that due to the adiabaticity condition (2) the system trajectories are located in the $\bar{M}$-space near the hyperplane of the $\bar{\mu}$-space. For example, the values $\mu_j^l$ should be of the order $\varepsilon$. Hence, the trajectory of the system’s representative point in the plane shown in Fig. 1 in fact is attached to the band $\mu_j^l \in (-\varepsilon, \varepsilon)$ and cannot move far away from it. Similarly, the dimensions of the higher-order derivatives $\mu_j^{(l)}$ are attached to even narrower bands of width $\varepsilon^l$.

Let us turn now to the proof of Theorem I (see Ref. 1). It is clear that the nonperturbed realization of the system used there (an arbitrary point in the considered region of $\bar{M}$-space) is located in the hyperplane of the $\bar{\mu}$-space. The perturbed realization of the system is represented by an arbitrary closed contour around the point considered whose projections are tied to bands $\mu_j^{(l)} \in (-\varepsilon^l, \varepsilon^l)$ in the planes $\mu_j^{(l)} = \mu_j^{(l-1)}$. The admissible closed contour satisfying the conditions (B1) must intersect with the hyperplane $\mu_j^{(l)} = 0$ (see Fig. 1), and hence the height of its projection on the plane $\mu_j^{(l)} = \mu_j^{(l-1)}$ is of order $\varepsilon^l$. In terms of $\mu_j$ dimensions, no restrictions on the contour geometry are posed, so it can always be chosen arbitrarily small. Meanwhile, in terms of $\mu_j^l$ dimensions its value is generally bounded both from above and from below by the order $\varepsilon$. This could prevent us from selecting a sufficiently small arbitrary contour for the perturbed system realization. However, all the admissible contours can be made arbitrarily small in the dimensions $\mu_j^{(l)}$, $l \geq 1$ by choosing a sufficiently small value of $\varepsilon$. Thus, it only remains to choose the necessarily small shape of the contour in the $\bar{\mu}$-space; its projections onto other dimensions will always be sufficiently small in order to use the theorem on the strong stability of Hamiltonian systems with periodic coefficients. Evidently, the Hamiltonian nature as well as the presence of generalized geometric amplitudes in a system does not depend on the magnitude of $\varepsilon$. So the proof remains valid for any small $\varepsilon$.

The proof of Theorem II does not face any additional difficulties as compared with the one given in Ref. 1. As for the proof of Theorem III, it involves, as in the case of Theorem I, the trajectory of the unperturbed realization of the system, as well as the closely lying trajectory of a perturbed realization, spiraling around the former one. The problem of existence of a sufficiently close perturbed trajectory can be resolved if we follow the logic pattern of the previous paragraph. Any admissible trajectory can be made arbitrarily close (in terms of dimensions $\mu_j^{(l)}$, $l \geq 1$) to the initial unperturbed one by choosing a sufficiently small value of $\varepsilon$, then utilizing the fact that the properties in question do not depend on the magnitude of the small $\varepsilon$.

In conclusion, let us note one more interesting point. The theory of Berry’s phase is remarkable in that the phase growth in the solution is determined only by the shape of the trajectory of the system’s representative point in the $\bar{\mu}$-space, rather than by the explicit time dependence of the parameters. A single trajectory may correspond to an infinitely large number of various realizations of the system. When considering the system trajectories in the generalized $\bar{M}$-space, the freedom in recovering the system’s realization (i.e., the $\bar{\mu}(\varepsilon t)$ dependence) is substantially restricted. Indeed, not only the sequence of variations in $\bar{\mu}$ values is specified, but also their derivatives up to the $(k-1)$th order. Still, a single contour corresponds as yet to an infinitely large number of realizations. For example, a closed contour in the $\bar{M}$-space does not necessarily correspond to periodic variations of the parameters: it is quite possible that the contour would be open in further dimensions $\bar{\mu}^{(k)}(\varepsilon t)$. However, in the infinite-dimensional space $\bar{M}^{(\infty)} = (\bar{\mu}(\varepsilon t), \bar{\mu}'(\varepsilon t), \ldots, \bar{\mu}^{(k)}(\varepsilon t), \ldots)$, in which all the parameter derivatives are taken into account, the system realization can be recovered unambiguously (if, say, the function $\bar{\mu}(\varepsilon t)$ is analytical) from knowledge of a contour in that space. Closed trajectories in that space correspond to periodic variations in $\bar{\mu}(\varepsilon t)$ values. Moreover, an analytical function $\bar{\mu}(\varepsilon t)$, as well as the corresponding trajectories, can be unambiguously reconstructed from one point on this trajectory. It is readily
seen that the coordinates of this point determine all the coefficients of the Taylor series. The $\tilde{M}^{(n)}$-space can be regarded as a certain representation of the space of analytic functions. Here every function corresponds to a certain trajectory, and conversely: each point in the $\tilde{M}^{(n)}$-space corresponds to a unique trajectory containing this point and a function.

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