Full length article

Elliptically polarized spatial solitons in cubic gyrotropic materials

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Received 8 February 1996; revised version received 7 June 1996; accepted 10 June 1996

Abstract

The problem of soliton propagation in nonlinear Kerr medium with linear optical activity and cubic anisotropy is considered. It is shown that balance between the nonlinearity and linear gyrotropy results in the existence of spatial elliptically polarized solitons with fixed state of polarization. For these novel stationary solitons, the bifurcation diagrams are constructed and preliminary stability analysis is carried out. The evolution of the state of polarization of the solitons nearby the stationary states is described using the Stokes parameters.

1. Introduction

Previously, a wide class of phenomena related to propagation of composite (vector) solitons has been considered in the literature [1–7]. The dynamics of vector solitons in media with natural or induced linear optical activity has attracted researcher’s attention recently [8–10]. It is known that the plane of polarization of linearly polarized light rotates during the propagation through the linear gyrotropic medium (linear optical activity) [11]. In the case of elliptically polarized light the natural gyrotropy leads to the rotation of the polarization ellipse. On the other hand, the plane of polarization of an elliptically polarized strong wave also undergoes the rotation in Kerr nonlinear isotropic media (nonlinear optical activity) [12,13]. The influence of these effects on the polarization evolution of “vector” solitons is of great interest.

It has been shown previously [14] that in the presence of linear and nonlinear competing mechanisms of polarization rotation in an isotropic medium, novel elliptically polarized temporal solitons exist which can propagate without rotation of the axes of polarization. These new soliton states influence the whole dynamics of soliton-like pulse propagation in nonlinear media with natural optical activity. The aim of the present work is the generalization of the problem of competing rotational mechanisms to the spatial soliton propagation in anisotropic media with linear optical activity. We show, analytically and numerically, that two new stationary vector solitons appear as a result of bifurcations from the fundamental solutions of generalized coupled nonlinear
Schrödinger equations. As a particular case, we consider cubic crystals. However, the results of this work can be extended to a medium of arbitrary symmetry.

The problem of soliton propagation can be complicated due to the presence of radiation caused by the oscillatory behavior of the soliton [15]. However, stationary solitons preserve a certain state of polarization at each point of the profile and the Stokes vector formalism [15] is a good tool to analyse the polarization dynamics of solitons. The method reduces the phase space of the problem to the so-called Poincaré sphere [16] and allows us to present approximate polarization dynamics with the beam propagation. Another useful tool for qualitative analysis is the Hamiltonian, $H$, versus energy, $Q$, diagram for stationary solutions. This plot allows us to predict stability properties of soliton states and, in some extent, their dynamics in the presence of radiation [15]. Namely, small radiation effects adiabatically reduce the values of energy as well as the Hamiltonian, which allows us to describe the direction of the evolution. We used both techniques to present the results and have shown the correspondence between different branches of solutions on the $\{H,Q\}$ diagram and trajectories on the Poincaré sphere.

2. Statement of the problem

The vector wave equation obtained from the Maxwell equations in the case of a nonmagnetic medium can be written in the form

$$\nabla^2 E - \frac{1}{c^2} \frac{\partial^2 D}{\partial t^2} - \frac{1}{c^2} \frac{\partial^2 P^{(NL)}}{\partial t^2} = 0. \tag{1}$$

An electric field of the light wave propagating along the $z$ axis in the slowly-varying-envelope approximation can be written as

$$E(r,t) = \frac{1}{2} \left( e_x E_x + e_y E_y \right) \exp(ik_0z - \omega t) + c.c. \tag{2}$$

Complex amplitudes $E_x(x,y,z)$ and $E_y(x,y,z)$ are slowly varying with $z$. We assume the medium to be cubic, lossless and gyrotropic so that the relation between the electric flux density $\mathbf{D}$ and the electric field $\mathbf{E}$ (2) is of the form

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + i\eta \sqrt{\varepsilon_0} \left[ \mathbf{E} \mathbf{E}^* \right]. \tag{3}$$

where $\varepsilon_0$ is the dielectric constant, $\eta$ is the diagonal component of the gyration tensor [11] and $e_z$ is the unit vector along the $z$ axis. Note that a relation of this form is common for isotropic and cubic materials (classes 432 or 23) with either natural or induced linear optical activity [11]. The last case is the most important because the degree of gyrotropy induced by an external magnetic field can be controlled externally.

Components of the nonlinear polarization vector $P^{(NL)}$ in the cubic Kerr medium are determined by the third-order susceptibility tensor $\chi^{(3)}$ so that in Cartesian coordinates we have

$$P^{(NL)}_i = 3\chi^{(3)}_{1111}E_i|E|^2 + 2\left(\chi^{(3)}_{1122} + \chi^{(3)}_{1212} + \chi^{(3)}_{1221}\right)E_i|E|^2 + \left(\chi^{(3)}_{1122} + \chi^{(3)}_{1212} + \chi^{(3)}_{1221}\right)E_jE_jE^*_i, \tag{4}$$

where $\chi^{(3)}_{abcd}$ are nonzero components of $\chi^{(3)}$ and the indices take the values $\{x,y\}$. The nonlinear polarization (4) can lead to rotation of the polarization ellipse of the plane electromagnetic wave [12] which we call "nonlinear gyrotropy" or "nonlinear optical activity".

As one can see from expression (3), the presence of linear gyrotropy leads to the difference in the refractive index $n$ for the right-(+) and left-hand (-) circularly polarized waves [11], so that

$$n^2_{\pm} = \varepsilon_{\pm} = \varepsilon^{(0)} \pm \eta \sqrt{\varepsilon^{(0)}}. \tag{5}$$

It is convenient to represent the electric field of the wave with a given polarization propagating in gyrotropic media as a sum of two circularly polarized components. In this case, the relation between the linear response
$D$ and the electric field $E$ has a simple form and all the effects occurring from the gyrotropy can be easily described. For right- and left-hand polarized components we have $E_+ \equiv U = E_x + i E_y$ and $E_- \equiv V = E_x - i E_y$.

For the slowly varying envelopes of the two circularly polarized components of the electric field, $U$ and $V$, we obtain the system of two coupled nonlinear Schrödinger equations which describe beam propagation in a cubic nonlinear Kerr medium with linear optical activity in the weak guidance approximation,

$$iU_\xi + \beta U + \frac{1}{2} U_{\tau \tau} + (|U|^2 + |A|^2) U + BU^* V^2 = 0,$$
$$iV_\xi - \beta V + \frac{1}{2} V_{\tau \tau} + (|U|^2 + |V|^2) V + BV^* U^2 = 0,$$

where we defined

$$\beta = \eta/\kappa,$$
$$A = 2, \quad B = 3(\chi_{1111} - \chi_{1122} - \chi_{1212} - \chi_{1221})/N,$$

and

$$N = 3\chi_{1111} + \chi_{1122} + \chi_{1212} + \chi_{1221}, \quad \kappa = \frac{3}{8} \frac{\omega^2 N}{c^2 \varepsilon_0}.$$  

The normalized longitudinal coordinate $\xi$ in (6) is related to the variable $z$ as follows: $\xi = (c \kappa/2 \omega) z$. The normalized transverse coordinate is denoted by $\tau = t \sqrt{\kappa}$. The terms with coefficients $A$ and $B$ can be referred as cross-phase modulation and four-wave mixing (energy exchange) terms, respectively. In contrast to isotropic media where $B = 0$, the coefficient $B$ in cubic media is nonzero and can take values independent of $A$. As will be shown later, this fact leads to the appearance of new soliton states with nonrotating polarization.

It should be noted that $U$ and $V$ in Eqs. (6) are circularly polarized components. Transformation to linearly polarized components, $E_x$ and $E_y$, will change the values of $A$ and $B$. The four-wave mixing term in (6) describes the energy exchange between the circular components. This process is not equivalent to the analogous one between linear components and results in completely different physics. For example, in the case of an isotropic medium ($B = 0$), there is no energy exchange between the circularly polarized components. However, the energy flow from one linearly polarized component of the field to another one still can be nonzero.

The total energy of the beam is invariant,

$$Q = \int_{-\infty}^{\infty} (|U|^2 + |V|^2) \, d\tau.$$  

The second integral of system (6), which is important for further consideration of the solutions and their stability, is the Hamiltonian

$$H = \int_{-\infty}^{\infty} \left[ -\beta (|U|^2 - |V|^2) + \frac{1}{2} (|U_\tau|^2 + |V_\tau|^2) - \frac{1}{2} (|U|^4 + |V|^4) - A|U|^2 |V|^2 - \frac{1}{2} B(U^2 V^* + U^* V^2) \right] \, d\tau.$$  

Using the above set of equations we can develop the analytical and numerical approaches allowing us to find the soliton solutions.

3. Stationary solutions

The soliton solutions of (6) can be written in the form

$$U = u(\tau, q) \exp(i q \xi), \quad V = v(\tau, q) \exp(i q \xi),$$

(12)
where the parameter of solution \( q \) is the intensity-dependent shift of the wavenumber, which is common for the two components due to the presence of energy flow between them. The wavevector mismatch caused by the linear optical activity is taken into account by an explicit gyrotropic factor \( \beta \).

For the new amplitudes \( u \) and \( v \), the set of equations (6) is transformed into

\[
\begin{align*}
\frac{1}{2}u_{\tau \tau} - (q - \beta)u + \frac{1}{2}v_{\tau \tau} + (|u|^2 + A|v|^2)u + Bu^2v^2 &= 0, \\
\frac{1}{2}v_{\tau \tau} - (q + \beta)v + \frac{1}{2}u_{\tau \tau} + (A|u|^2 + |v|^2)v + Bv^2u^2 &= 0.
\end{align*}
\]

(13)

We are interested in stationary solutions of (13) which do not depend on \( \xi \) and therefore have the form

\[
u = u_0(\tau) \exp(\imath\varphi_1), \quad v = v_0(\tau) \exp(\imath\varphi_2),
\]

(14)

where the phase difference \( \Delta \varphi = \varphi_1 - \varphi_2 \) can take the values \( \Delta \varphi = \pm \pi \) or \( \Delta \varphi = \pm \pi/2 \). The shapes of the solitary waves \( u_0 \) and \( v_0 \) can be found from the set of two second-order ordinary differential equations,

\[
\begin{align*}
\frac{1}{2}u_{\tau \tau} - (q - \beta)u + (|u|^2 + A|\pm B|^2|v|^2)u &= 0, \\
\frac{1}{2}v_{\tau \tau} - (q + \beta)v + (A|u|^2 + |v|^2)v &= 0,
\end{align*}
\]

(15)

where the stationary profiles \( u_0 \) and \( v_0 \) are real functions subject to the boundary conditions: \( u_0, v_0 \to 0 \) and \( u_{\tau \tau}, v_{\tau \tau} \to 0 \) when \( \tau \to \pm \infty \). In these equations we should choose the plus sign for \( \Delta \varphi = \pm \pi \) and the minus sign for \( \Delta \varphi = \pm \pi/2 \).

Eqs. (13) have simple exact solutions representing bright solitons with right- and left-hand circular polarizations. The stationary profiles for these two solitons have the following forms,

\[
\begin{align*}
u_0 &= \frac{\sqrt{2(q - \beta)}}{\cosh(\sqrt{2(q - \beta)}\tau)}, \quad v_0 = 0, \quad \text{for right-hand circular polarization,} \\
u_0 &= 0, \quad v_0 = \frac{\sqrt{2(q + \beta)}}{\cosh(\sqrt{2(q + \beta)}\tau)}, \quad \text{for left-hand circular polarization.}
\end{align*}
\]

(16)

(17)

respectively.

All stationary soliton solutions can be conveniently presented as curves in the space of variables \( \{q, Q, H\} \). The curves corresponding to the fundamental solutions are shown on the “energy-dispersion” (\( \{Q, q\} \)) and “Hamiltonian-energy” (\( \{H, Q\} \)) diagrams (Fig. 1). They are given by

\[
\begin{align*}
Q_u(q) &= 2\sqrt{2(q - \beta)}, \quad Q_v(q) &= 2\sqrt{2(q + \beta)},
\end{align*}
\]

(18)

and

\[
\begin{align*}
H_u(Q) &= -\frac{1}{24}Q_u^3 - \beta Q_u, \quad H_v(Q) &= -\frac{1}{24}Q_v^3 + \beta Q_v.
\end{align*}
\]

(19)

The stationary solutions corresponding to elliptically polarized pulses (light curves in Fig. 1) can be found numerically. These solutions appear as a result of bifurcation from the circularly polarized solutions in the space of variables \( \{q, Q, H\} \). The point of bifurcation can be found explicitly in terms of the parameter \( q \). Then we can obtain the energy, \( Q(q) \), and Hamiltonian, \( H(Q(q)) \), at the point of bifurcation. Using the perturbation analysis, we can also find approximate analytical expressions for elliptically polarized solutions near the bifurcations points.

Let us suppose that the solution is “nearly circular” so that the contribution of the second component to the beam is very small, say \( v_0/u_0 = \varepsilon \ll 1 \). In zeroth order we have

\[
\frac{1}{2}u_{\tau \tau} - u_0(q - \beta) + u_0|u_0|^2 = 0,
\]

(20)
Fig. 1. Energy-dispersion and Hamiltonian versus energy diagrams for the cases (a) $-1 < B < 1$, $\beta = 1.0$; (b) $B < -1$, $\beta = 1.0$; (c) $B < -17/8$, $\beta = 1.0$. Black solid curves correspond to circularly polarized solitons. The branches of elliptically polarized solutions with $\pm \pi$ phase shift between polarization components and the branches of out-of-phase solutions are marked by numbers 1 and 2, correspondingly.

which has a trivial sech-form solution. Omitting the terms of the order of $\varepsilon^2$ and higher, we can obtain the self-consistent equation for $u_0$,

$$
\frac{1}{2} u_{0rr} + \left[ -(q + \beta) + \frac{2(q - \beta)}{\text{sech}^2(\sqrt{2}(q - \beta)\tau)} \right] u_0 = 0.
$$

(21)

The solution of Eq. (20) for $u_0$ and the linearized equation (21) for $u_0$ allows us to find bifurcations in the space of variables \{q, Q, H\} (see Fig. 1). The number and the type of nontrivial solutions depend on the parameters of the medium $A$ and $B$. Since, in our case, the value of the parameter $A$ is fixed, $A = 2$, the classification of bifurcations is defined by the value of the parameter $B$ only:

(a) In the case when $-1 < B < 1$, there are two bifurcation points. Both branches of the new stationary elliptically polarized solutions split off from the circularly polarized solitons: $u_0 = 0$, $u_0 \neq 0$ at the points
of bifurcations which are given by the critical parameters $q_+$ and $q_-,$

$$q_+ = \beta \frac{\nu_+^2 + 1}{[\nu_+^2 - 1]^2}, \quad q_- = \beta \frac{\nu_-^2 + 1}{[\nu_-^2 - 1]^2},$$  \hspace{1cm} (22)

where $\nu_+$ and $\nu_-$ are defined by the condition of existence of the localized state in the problem (21) [17],

$$\nu_{\pm} = \frac{1}{2} \left( \sqrt{8(2 \pm B)} + 1 - 1 \right).$$  \hspace{1cm} (23)

The values of the energy and Hamiltonian at the points of bifurcation can be found substituting $q_+$ and $q_-$ into expressions (18), (19). At the point $\{q_+, Q_u(q_+), H_u(q_+)\},$ the solutions with phase difference between the partial components $\Delta \varphi = \pm \pi$ appear. The elliptically polarized solutions with $\Delta \varphi = \pm \pi/2$ correspond to the second point of bifurcation $\{q_-, Q_u(q_-), H_u(q_-)\}.$ At each point of bifurcation, we formally obtain two new solutions which differ from each other by the sign of the phase shift between the circularly polarized components. As will be clear from the analysis of the polarization evolution, these two solutions are physically equivalent.

(b) If $B < -1,$ then two stationary elliptically polarized solutions split off from two different branches of circularly polarized solitons. The bifurcation points in this case are $\{q_+, Q_u(q_+), H_u(q_+)\}$ for the solutions with $\Delta \varphi = \pm \pi$ starting from the branch $u_0 = 0, \nu_0 \neq 0,$ and $\{q_-, Q_u(q_-), H_u(q_-)\}$ for the solutions with $\Delta \varphi = \pm \pi/2$ starting from the branch $\nu_0 = 0, u_0 \neq 0.$

(c) In the case of $B > 1,$ two stationary elliptically polarized solutions also split off from two different branches of circularly polarized solitons. The bifurcation points are $\{q_+, Q_u(q_+), H_u(q_+)\}$ for the solutions with $\Delta \varphi = \pm \pi$ starting from the branch $u_0 = 0, u_0 \neq 0,$ and $\{q_-, Q_u(q_-), H_u(q_-)\}$ for the solutions with $\Delta \varphi = \pm \pi/2$ starting from the branch $u_0 = 0, \nu_0 \neq 0.$

(d) According to condition (23), in the case when $B \leq -17/8$ or $B \geq 17/8,$ there is only one bifurcation point corresponding to two degenerate stationary elliptically polarized solutions in addition to the fundamental ones. Bifurcations at different values of the parameter $B$ in this case can be considered in the same way as for the elliptically polarized solitons in a linearly birefringent medium [15].

A summary of this classification is the following: if $A$ and $B$ are independent, there are two points of bifurcation, where two pairs of solutions with elliptic polarization split off from the solutions with circular polarization. These solutions exist only when the beam energy $Q$ is above a certain limit. This threshold can be found analytically. The bifurcation diagrams, $\{Q, q\}$ and $\{H, Q\}$ (Fig. 1), show that new solutions can start from the same (c) or different branches (a), (b).

4. Evolution of the state of polarization

All possible states of polarization of the soliton solution of (6), in analogy with a quasi-monochromatic plane wave, can be represented by a set of four Stokes parameters [16]. In terms of circular components of the electric field, the Stokes parameters, denoted by $s_0, s_1, s_2,$ and $s_3,$ are defined as follows:

$$s_0 = 2(|U|^2 + |V|^2), \quad s_1 = 2(U^*V + UV^*), \quad s_2 = -2i(U^*V - UV^*), \quad s_3 = 2(|U|^2 - |V|^2).$$  \hspace{1cm} (24)

These four parameters are real functions of $\xi$ and $\tau.$ They vary along the propagation coordinate as well as across the beams, but for beams with a certain state of polarization they satisfy the following condition:

$$s_0^2 = s_1^2 + s_2^2 + s_3^2.$$  \hspace{1cm} (25)

The state of polarization of the soliton solution of (6) can be represented by a point on the Poincaré sphere [16] in the space $\{s_1, s_2, s_3\}.$
Fig. 2. Different orientations of the polarization ellipse for the solutions located on different points of the Poincaré sphere.

Let us now consider the evolution of the state of the polarization in terms of Stokes parameters. The equations for the complex amplitudes \( u \) and \( v \) can now be rewritten as an equivalent system for four real values,

\[
\frac{d}{d\xi} s_0 = 0, \quad \frac{d}{d\xi} s_1 = -2\beta s_2 + \frac{1 - B}{2} s_3 s_0, \quad \frac{d}{d\xi} s_2 = 2\beta s_1 - \frac{1 + B}{2} s_1 s_3, \quad \frac{d}{d\xi} s_3 = 2Bs_1 s_2.
\]  
(26)

Note that the parameters (24) are differential ones. It means that they describe the evolution of the state of polarization at each point \( \tau \) of the stationary profile \( f(\tau) = \sqrt{u^2 + v^2} \) of the beam. The state of polarization at any given point of the soliton profile changes during the beam propagation along the \( \xi \)-axis. Its evolution with \( \xi \) is described by Eqs. (26).

Approximately, a certain state of polarization at a given point of the profile can be applied to the soliton-like beam as a whole [18]. For the stationary soliton-like solutions found numerically, the degree of ellipticity (see Fig. 2) of the beam \( \theta(\tau) = \arcsin(s_3/s_0) \) changes from its top \( (\tau = 0) \) to the tails \( (\tau \to \pm \infty) \) only slightly. This gives great advantage in the analysis of polarization dynamics allowing to use the approximation of averaged profile [15]. Namely, although the solution of (6) cannot be exactly separable due to the energy exchange between the two components of the soliton, we can assume the \( \tau \)-dependent profile of both components to be equal and then follow the evolution of the amplitudes depending only on the longitudinal coordinate \( \xi \),

\[
u = f(\tau) x(\xi) \exp(i\varphi_1), \quad \nu = f(\tau) y(\xi) \exp(i\varphi_2),
\]  
(27)

where the functions \( x(\xi) \) and \( y(\xi) \) are real. Using the above assumption we can transform the differential Stokes parameters into the integral ones (relating to the whole profile of a beam as a unit) using the definition

\[
S_i(\xi) = \frac{\int s_i(\tau, \xi) \, d\tau}{\int f^2(\tau) \, d\tau},
\]  
(28)

so that

\[
S_0 = 2(x^2 + y^2), \quad S_1 = 4xy \cos \Delta \varphi, \quad S_2 = 4xy \sin \Delta \varphi, \quad S_3 = 2(x^2 - y^2).
\]  
(29)

Now the evolution of the state of polarization of any soliton-like solution of (6) during propagation can be qualitatively analyzed as a motion of the Stokes vector \( S = \{S_1, S_2, S_3\} \) on the Poincaré sphere. This evolution is governed by the following set of first-order differential equations:

\[
\frac{d}{d\xi} s_0 = 0, \quad \frac{d}{d\xi} s_1 = -2\beta s_2 + (1 - B) g S_2 S_3, \quad \frac{d}{d\xi} s_2 = 2\beta s_1 - (1 + B) g S_1 S_3, \quad \frac{d}{d\xi} s_3 = 2Bg S_1 S_2,
\]  
(30)
where we have introduced the parameter \( g = \frac{1}{2} \int_{-\infty}^{\infty} f^4(\tau) \, d\tau / \int_{-\infty}^{\infty} f^2(\tau) \, d\tau \). This set of equations has two integrals, one of which, \( W_1 = S_0^2 \), comes from the conservation of the total energy of the beam, and another one is

\[
W_2 = -g(1 + B)S_1^2 - g(1 - B)S_2^2 - 4\beta S_3. \tag{31}
\]

The presence of these two integrals allows us to write down the exact solution of this system in terms of elliptical Jacobi functions which is presented in the Appendix.

If the beam exactly conserves its profile \( f(\tau) \) during propagation then the value \( s_0 \) is identical to the value \( 2f^2(\tau) \) and therefore \( S_0 = 2 \). If this is so, the only parameter of the family of trajectories on the Poincaré sphere is the parameter \( g \), implicitly depending on the beam energy \( Q \). An approximation of the averaged profile permits us to establish the qualitative relation between the parameters \( g \) and \( q \). Namely, taking a profile of the form \( f(\tau) = \sqrt{2q}/\sech(\sqrt{2q}\tau) \), which corresponds to the average of the exact circularly polarized solutions (16) at \( \beta = 0 \), we obtain \( g = 2q/3 \).

In order to analyse the trajectories of the Stokes vector on the Poincaré sphere, we start from the classification of the fixed points \( (dS_i/d\xi = 0 \text{ for } i = 1, 2, 3) \) of system (30). These points correspond to the stationary soliton-like solutions of Eqs. (6) and since the system is Hamiltonian, they can only be elliptical or saddle-type which corresponds to the stable or unstable behaviour of stationary solutions. The main feature of these solutions is that the axes of the polarization ellipse do not change their length and orientation with the propagation. Now, consider the fixed points within the frame of our classification of bifurcations (see Section 3).

First of all, let us note that the stationary points on the two poles of the Poincaré sphere \( \{0, 0, S_0\} \) and \( \{0, 0, -S_0\} \) correspond to the stationary solutions of (6) with counter-rotating circular polarizations \( \{u \neq 0, v = 0\} \) and \( \{v \neq 0, u = 0\} \) and do exist at any values of the parameters. In addition to these two, there can exist four stationary solutions with elliptical polarization and nonrotating axes of the polarization ellipse. The conditions for their existence essentially depend on the parameter \( g \) which has two critical values,

\[
g_+ = \left| \frac{\beta}{-1 + B} \right|, \quad g_- = \left| \frac{\beta}{-1 - B} \right|. \tag{32}
\]

Two fixed points on the Poincaré sphere corresponding to the beams with \( \pm \pi/2 \) phase-shifted components appear when \( g \geq g_+ \) and have the coordinates \( \{S_1 = 0, S_2 = \pm 2\sqrt{1 - (g_+/g)^2}, S_3 = -2g_+/g\} \). Another two fixed points corresponding to the solutions with \( \Delta \varphi = \pm \pi \) can exist only if \( g \geq g_- \) and have the locations \( \{S_1 = \pm 2\sqrt{1 - (g_-/g)^2}, S_2 = 0, S_3 = -2g_-/g\} \). Whether these points are located above or below the equatorial line depends on particular values of the parameters \( A \) and \( B \) and, therefore, on the sign of the quantities \( g_+ \) and \( g_- \). It corresponds to the right- or left-hand elliptically polarized states. Obviously, if the branches representing these new solutions on the \( \{H, Q\} \) and \( \{Q, q\} \) diagrams split off from the same circularly polarized solution then the fixed points are located in the same hemisphere. They lie in different hemispheres in the opposite case. Fig. 2 gives the idea about the polarization state of the beam for the solutions located at different points on the sphere. Any two solutions which are distinguished only by the sign of the phase shift, correspond to the stationary soliton with the same determined fixed orientation of the axes of the polarization ellipse. Hence, they are physically identical.

(a) In the case when \(-1 < B < 1\), four additional stationary solutions with elliptical polarization split off from the solution \( v_0 = 0, u_0 \neq 0 \), and are represented by four fixed points located above the equator of the Poincaré sphere \( (S_3 > 0) \) (see Fig. 3a). The state of polarization is right-hand elliptic.

(b) In the case \( B < -1 \), two fixed points corresponding to the solutions with \( \Delta \varphi = \pm \pi/2 \) are located above the equatorial line \( (S_3 > 0) \). The pair of points corresponding to stationary elliptically polarized solutions with \( \Delta \varphi = \pm \pi \), starting from the branch \( u_0 = 0, v_0 \neq 0 \), lies under the equatorial line \( (S_3 < 0) \) (see Fig. 3b).
In the case of $B > 1$ and $B < -1$, the stationary elliptically polarized solutions also split off from the two different branches of the fundamental solutions. Two pairs of corresponding fixed points on the Poincaré sphere are located above and under the equatorial line. Namely, the pair corresponding to the solutions with the phase difference $\Delta \varphi = \pm \pi / 2$ has coordinates with $S_3 < 0$. Another pair corresponding to the solutions with $\Delta \varphi = \pm \pi$ has the location in $S_3 > 0$. This case is complimentary to the previous one.

When $B = 1$ or $B = -1$, only one bifurcation point exists. It corresponds to the two degenerated stationary elliptically polarized solutions shown in Fig. 3c.

The existence of four nontrivial elliptically polarized stationary solutions in this system is possible due to the presence of linear gyrotropy. When the gyrotropy parameter $\beta$ turns to zero, we still have six fixed points on the Poincaré sphere: two points corresponding to circularly polarized solutions, $\{0, 0, \pm S_0\}$, and four points located on the equatorial line ($\{\pm S_0, 0, 0\}$, $\{0, 0, \pm S_0\}$) and corresponding to the stationary solutions with linear polarization. This case concerns pure nonlinear optical activity which does not lead to rotation of the plane of polarization for linearly polarized beams. In the last case, any point of the equatorial line becomes a fixed point.

5. Discussion of stability

It follows from the canonical form of Eqs. (6) that solutions with minimal Hamiltonian at any fixed energy $Q$ are stable [19]. This means that vector solitons corresponding to lowest branches on the $\{H, Q\}$ diagram are stable (see Fig. 1). Hence, the right-hand circularly polarized solitons are stable up to the point of bifurcation. Beyond the bifurcation point, new elliptically polarized solitons, which have $\pm \pi$ phase-shift between the components of polarization, appear and the circularly polarized solution loses its stability. The value of the Hamiltonian for the new solution is the lowest at any value of $Q$, which means that they must be stable. All soliton states represented by upper branches on the $\{H, Q\}$ diagram are unstable. However, the mechanisms of instability can be two-fold. Firstly, the solution is obviously unstable if the corresponding fixed point on the Poincaré sphere is of saddle type (strong instability).

Another source of instability in the system is radiation from the leading beam (weak instability). The beam decreases its energy with propagation, and therefore the values $Q$ and $H$ also decrease. In principle, any initial state, which does not correspond to the absolute minimum of the Hamiltonian, can converge to the stable state after emitting some amount of radiation [15]. The convergence of the corresponding trajectories to the fixed points on the Poincaré sphere can be investigated by the numerical solution of the original equations (6).

Our numerical simulations are based on the technique described in Ref. [15]. It appears that even for the case of strong radiation emission, our approximation describes the structure of trajectories on the Poincaré sphere correctly. The plot of the exact trajectories for the case of strong radiation is presented in Fig. 4. We can see
that different input beams transform into the stationary solution with the lowest Hamiltonian very quickly. Even a small perturbation added to the initial circularly polarized state (saddle point on the pole of the Poincaré sphere) tends to grow rapidly, which confirms the instability of this solution. The detailed discussion of the mechanism of instability and the calculations of the perturbation growth rates remain to be presented.

In conclusion, we have found the stationary elliptically polarized soliton solutions for beams in Kerr media with linear optical activity and cubic anisotropy. We used a combined analysis based on energy-dispersion, the Hamiltonian versus energy diagram and Poincaré sphere formalism to investigate the stability of stationary solutions and the dynamics of the state of polarization of the solitons.

Acknowledgements

The work was supported by the Australian Photonics Cooperative Research Center (APCRC). N.N.A. is grateful to Professor A. Snyder for fruitful and substantial discussions.

Appendix A

The bifurcations pattern in the system and the trajectories on the Poincaré sphere are different in two cases: (i) $|B| > 1$ and (ii) $|B| < 1$.

(i) In the case when $|B| > 1$, three solutions cover up the whole family of trajectories: $S^{(1)} = \{S_1^{(1)}, S_2^{(1)}, S_3^{(1)}\}$, where

$$S_1^{(1)} = \pm \sqrt{\frac{|B| - 1}{2|B|} \frac{\sqrt{(Q_2 - Q_4)(Q_1 - Q_3) - (Q_2 - Q_3)(Q_1 - Q_4) \sn^2[C\xi, k]}}{(Q_2 - Q_4)^{-1}(Q_3 - Q_4)^{-1}(Q_2 - Q_4 - (Q_2 - Q_3) \sn^2[C\xi, k])}},$$

$$S_2^{(1)} = \pm \sqrt{\frac{|B| + 1}{2|B|} \frac{(Q_2 - Q_3) \sqrt{(Q_2 - Q_4)(Q_3 - Q_4) \sn^2[C\xi, k] (1 + \sn^2[C\xi, k])}}{Q_2 - Q_4 - (Q_2 - Q_3) \sn^2[C\xi, k]}}.$$
\[ S_3^{(1)} = \frac{Q_3(Q_2 - Q_4) + Q_4(Q_3 - Q_2) \, \text{sn}^2[C \xi, k]}{Q_2 - Q_4 - (Q_2 - Q_3) \, \text{sn}^2[C \xi, k]}, \]  

(A.1)

\[ S_2^{(2)} = \{ S_1^{(2)}, S_2^{(2)}, S_3^{(2)} \}, \]

where

\[ S_1^{(2)} = \pm \frac{\sqrt{B^2 - 1}}{2|B|} \frac{(Q_1 - Q_3)(Q_1 - Q_3) \, \text{sn}^2[C \xi, k] (1 + \text{sn}^2[C \xi, k])}{Q_1 - Q_5 - (Q_2 - Q_1) \, \text{sn}^2[C \xi, k]}, \]

\[ S_2^{(2)} = \pm \frac{\sqrt{B^2 + 1}}{2|B|} \frac{\sqrt{(Q_2 - Q_4)(Q_1 - Q_3) - (Q_2 - Q_3)(Q_1 - Q_4) \, \text{sn}^2[C \xi, k]}}{(Q_1 - Q_2)^{-1}(Q_1 - Q_3)^{-1}(Q_1 - Q_3 - (Q_2 - Q_3) \, \text{sn}^2[C \xi, k])}, \]

\[ S_3^{(2)} = \frac{Q_2(Q_1 - Q_3) + Q_1(Q_3 - Q_2) \, \text{sn}^2[C \xi, k]}{Q_1 - Q_3 - (Q_2 - Q_3) \, \text{sn}^2[C \xi, k]}, \]

(A.2)

and \[ S^{(3)} = \{ S_1^{(3)}, S_2^{(3)}, S_3^{(3)} \}, \]

where

\[ S_1^{(3)} = \frac{\sqrt{F_+} \pm \sqrt{F_-}}{2}, \]

\[ S_2^{(3)} = \frac{\sqrt{F_+} \pm \sqrt{F_-}}{2}, \]

\[ S_3^{(3)} = \frac{(Q_1 + Q_3)(Q_1 - Q_4)^2 + 2(Q_1 - Q_3)(Q_1 Q_2 - Q_1 Q_4) \, \text{sn}^2[C \xi, 1/k] - (Q_1 + Q_3)(Q_2 - Q_4)(Q_1 - Q_3) \, \text{sn}^2[C \xi, 1/k]}{2(Q_1 \, \text{cs}^2[C \xi, 1/k] + Q_3 \, \text{sn}^2[C \xi, 1/k] - Q_4)(Q_1 + Q_3)(Q_2 - Q_4)(1 + \text{sn}^2[C \xi, 1/k])}. \]

(A.3)

In Eq. (A.3) we defined

\[ F_\pm = S_0^2 - S_3^{(3)2} \pm \frac{1}{2 B_0^2} \frac{dS_3^{(3)}}{d\xi}. \]

(A.4)

Solutions \[ S^{(2)}, S^{(1)}, \] and \[ S^{(3)} \] exist when the inequality \[ S_0 > |2\beta/[g(1 \pm |B|)]| \] holds. In this case, two points with elliptical polarization appear in different hemispheres (Fig. 3b). They correspond to the beams with clockwise and counterclockwise polarization rotation. Trajectories \[ S^{(1)} \] are confined by the separatrix loop going through the point \{0, 0, S_0\}, trajectories \[ S^{(2)} \] lie inside the separatrix loop going through the point \{0, 0, -S_0\}. The trajectories beyond both separatrices are described by the solution \[ S^{(3)} \]. The modulus of the Jacobi function,

\[ k = \sqrt{\frac{(Q_2 - Q_4)(Q_1 - Q_4)}{(Q_1 - Q_3)(Q_2 - Q_4)}}, \]

(A.5)

is varying from 0 to 1, the parameter \( C \) is

\[ C = \frac{1}{2} g \sqrt{B^2 - 1} \sqrt{(Q_1 - Q_3)(Q_2 - Q_4)}. \]

(A.6)

The only parameter of the solutions for a given radius of the Poincaré sphere, \( W_1 \equiv S_0^2 \), is the modulus \( k \). The dependence of the second integral, \( W_2 \), on the modulus is given by the expression

\[ W_2 = \frac{4(k^2 - 1)[S_0^2 g^2(1 - B^2) + 4\beta^2] \pm \sqrt{\sigma}}{g[4(k^2 - 1) - B^2(k^2 - 2)]}, \]

(A.7)

where

\[ \sigma = 4B^2(k^2 - 2)^2[(S_0^2 g^2(1 - B^2) + 4\beta^2)^2 + 4g^2\beta^2 S_0^2 B^2 k^4]. \]

(A.8)

Here we should always use the sign “+” in (A.7) when referring to the solution (A.1), and the sign “−” when referring to the solution (A.2). The solution (A.3) is invariant relative to the sign in (A.7).
Parameters $Q_i$ take the forms

\begin{align*}
Q_1 &= 2\beta / \left[ g(1 \mp |B|) \right] + \sqrt{S_0^2 + W_2 / \left[ g(1 \mp |B|) \right]} + 4\beta^2 / \left[ g^2(1 \mp |B|)^2 \right], \\
Q_2 &= 2\beta / \left[ g(1 \pm |B|) \right] + \sqrt{S_0^2 + W_2 / \left[ g(1 \pm |B|) \right]} + 4\beta^2 / \left[ g^2(1 \pm |B|)^2 \right], \\
Q_3 &= 2\beta / \left[ g(1 \pm |B|) \right] - \sqrt{S_0^2 + W_2 / \left[ g(1 \pm |B|) \right]} + 4\beta^2 / \left[ g^2(1 \pm |B|)^2 \right], \\
Q_4 &= 2\beta / \left[ g(1 \mp |B|) \right] - \sqrt{S_0^2 + W_2 / \left[ g(1 \mp |B|) \right]} + 4\beta^2 / \left[ g^2(1 \mp |B|)^2 \right].
\end{align*}

(A.9)

Here, we should choose the upper sign for the solution $S^{(1)}$ and the lower sign for the solution $S^{(2)}$.

Two above written solutions, (A.1) and (A.2), describe the trajectories inside two separatrices on the sphere (see Fig. 3b). The separatrix solutions themselves can be obtained in the limit $k \to 1$. On the separatrix loop going through the point $\{0, 0, S_0\}$, the value of the second integral is $W_2 = -4\beta S_0$, and the Stokes vector components have the form

\begin{align*}
S_1^{(1)} &= \frac{\pm 2\sqrt{2\beta}}{g} \cosh[C\xi] \frac{(S_0 g(|B| - 1) + 2\beta) \sqrt{(S_0 g(|B| + 1) - 2\beta)}}{S_0 g(B^2 - 1) + 2\beta(B \cosh[2C\xi] + 1)}, \\
S_2^{(1)} &= \frac{\pm 2\sqrt{2\beta}}{g} \sinh[C\xi] \frac{(S_0 g(|B| - 1) - 2\beta) \sqrt{(S_0 g(|B| + 1) + 2\beta) \cosh[2C\xi]}}{S_0 g(B^2 - 1) + 2\beta(B \cosh[2C\xi] + 1)}, \\
S_3^{(1)} &= (4\beta - gS_0) - g|B|S_0 \frac{(2\beta - gS_0) \cosh(2C\xi) + g|B|S_0}{(2\beta - gS_0) + g|B|S_0 \cosh(2C\xi)},
\end{align*}

(A.10)

where $C = \sqrt{S_0^2 g^2(1 - B^2) + 4\beta^2 - 4\beta g S_0}$. In this limit, the second solution degenerates into the fixed point on the pole of the Poincaré sphere, $\{0, 0, -S_0\}$. On the other separatrix loop, the second integral takes the value $W_2 = 4\beta S_0$, and the components of the Stokes vector are

\begin{align*}
S_1^{(2)} &= \frac{\pm 2\sqrt{2\beta}}{g} \sinh[C\xi] \frac{(S_0 g(|B| - 1) - 2\beta) \sqrt{(S_0 g(|B| + 1) + 2\beta) \cosh[2C\xi]}}{S_0 g(B^2 - 1) + 2\beta(B \cosh[2C\xi] - 1)}, \\
S_2^{(2)} &= \frac{\pm 2\sqrt{2\beta}}{g} \cosh[C\xi] \frac{(S_0 g(|B| + 1) + 2\beta) \sqrt{(S_0 g(|B| - 1) - 2\beta)}}{S_0 g(B^2 - 1) + 2\beta(B \cosh[2C\xi] - 1)}, \\
S_3^{(2)} &= (4\beta + gS_0) + g|B|S_0 \frac{(2\beta + gS_0) \cosh(2C\xi) - g|B|S_0}{(2\beta + gS_0) - g|B|S_0 \cosh(2C\xi)},
\end{align*}

(A.11)

where $C = \sqrt{S_0^2 g^2(1 - B^2) + 4\beta^2 + 4\beta g S_0}$. The solution $S^{(1)}$, in this limit, leads to the fixed point $\{0, 0, S_0\}$. The solution (A.3) tends to the separatrices (A.10) and (A.11) with $W_2 = 4\beta S_0$ and $W_2 = -4\beta S_0$, correspondingly.

In the limit $k \to 0$, the solution $S^{(1)}$ gives a pair of fixed points corresponding to stationary elliptically polarized beams,

\begin{align*}
S_1^{(1)} &= \pm \sqrt{S_0^2 - 4\beta^2 / \left[ g^2(1 - |B|)^2 \right]}, \\
S_2^{(1)} &= 0, \\
S_3^{(1)} &= 2\beta / \left[ g(1 + |B|) \right],
\end{align*}

(A.12)

for which the second integral takes the value $W_2 = -g(1 + |B|)S_0 - 4\beta^2 / \left[ g(1 + |B|) \right]$. The solution $S^{(2)}$ gives another pair of fixed points for which the second integral takes the value $W_2 = -g(1 + |B|)S_0 - 4\beta^2 / \left[ g(1 + |B|) \right]$. 

\begin{align*}
S_1^{(2)} &= 0, \\
S_2^{(2)} &= \pm \sqrt{S_0^2 - 4\beta^2 / \left[ g^2(1 + |B|)^2 \right]}, \\
S_3^{(2)} &= 2\beta / \left[ g(1 - |B|) \right].
\end{align*}

(A.13)
(ii) In the case when $|B| < 1$, there are two solutions describing the family of the trajectories inside two separatrix loops. The trajectories $S^{(1)} = \{S_1^{(1)}, S_2^{(1)}, S_3^{(1)} \}$, where

$$
S_1^{(1)} = \pm \sqrt{\frac{1 - |B|}{2|B|}} (Q_2 - Q_4) \text{cn}[C\xi, k] \frac{\sqrt{(Q_1 - Q_3)(Q_1 - Q_2)}}{Q_2 - Q_4 + (Q_1 - Q_2) \text{sn}^2[C\xi, k]},
$$

$$
S_2^{(1)} = \pm \sqrt{\frac{1 + |B|}{2|B|}} (Q_1 - Q_4) \text{sn}[C\xi, k] \frac{\sqrt{(Q_2 - Q_4)(Q_1 - Q_2)}}{Q_2 - Q_4 + (Q_1 - Q_2) \text{sn}^2[C\xi, k]},
$$

$$
S_3^{(1)} = \frac{Q_1(Q_2 - Q_4) + Q_4(Q_1 - Q_2) \text{sn}^2[C\xi, k]}{Q_2 - Q_4 + (Q_1 - Q_2) \text{sn}^2[C\xi, k]},
$$

embrace the point $\{0, 0, S_0\}$. The other fixed point, on the pole of the Poincaré sphere, is embraced by the trajectories given by the solutions $S^{(2)} = \{S_1^{(2)}, S_2^{(2)}, S_3^{(2)} \}$, where

$$
S_1^{(2)} = \pm \sqrt{\frac{1 - |B|}{2|B|}} (Q_2 - Q_4) \text{cn}[C\xi, k] \frac{\sqrt{(Q_1 - Q_3)(Q_1 - Q_2)}}{Q_1 - Q_3 + (Q_3 - Q_4) \text{sn}^2[C\xi, k]},
$$

$$
S_2^{(2)} = \pm \sqrt{\frac{1 + |B|}{2|B|}} (Q_1 - Q_4) \text{sn}[C\xi, k] \frac{\sqrt{(Q_2 - Q_4)(Q_1 - Q_3)}}{Q_1 - Q_3 + (Q_3 - Q_4) \text{sn}^2[C\xi, k]},
$$

$$
S_3^{(2)} = \frac{Q_4(Q_1 - Q_3) + Q_1(Q_3 - Q_4) \text{sn}^2[C\xi, k]}{Q_1 - Q_3 + (Q_3 - Q_4) \text{sn}^2[C\xi, k]}.
$$

The modulus, $0 < k < 1$, defining the family of the trajectories, is given by the expression

$$
k = \sqrt{\frac{(Q_1 - Q_2)(Q_3 - Q_4)}{(Q_1 - Q_3)(Q_2 - Q_4)}},
$$

and the parameter $C$ takes the value

$$
C = \frac{1}{2} g \sqrt{1 - B^2} \sqrt{(Q_1 - Q_3)(Q_2 - Q_4)}.
$$

Solutions $S^{(2)}$ and $S^{(1)}$ exist if the inequality $S_0 > |2B/[g(1 \pm |B|)]|$ holds. Two points with elliptical polarization appear in the same (upper) hemisphere (Fig. 3a). The dependence of the second integral, $W_2$, on the modulus, $k$, is given by the expression

$$
W_2 = \frac{-4k^2 S^2 g^2 (1 - B^2) + 4\beta^2 \pm 2 \sqrt{\sigma}}{g(4k^2 - B^2(k^2 + 1)^2)},
$$

where

$$
\sigma = k^2 B^2[S^2 g^2 (1 - B^2) - 4\beta^2]^2 + 8g^2 B^2 \beta^2 S^2 (k^2 - 1)^2 (k^4 + 1).
$$

Here we should use plus for the solution $S^{(1)}$, and minus for the solution $S^{(2)}$. Parameters $Q_i$ take the form

$$
Q_1 = 2\beta / [g(1 + |B|)] + \sqrt{S^2 + W_2 / [g(1 + |B|)] + 4\beta^2 / [g^2(1 + |B|)^2]},
$$

$$
Q_2 = 2\beta / [g(1 - |B|)] + \sqrt{S^2 + W_2 / [g(1 - |B|)] + 4\beta^2 / [g^2(1 - |B|)^2]},
$$

$$
Q_3 = 2\beta / [g(1 - |B|)] - \sqrt{S^2 + W_2 / [g(1 - |B|)] + 4\beta^2 / [g^2(1 - |B|)^2]},
$$

$$
Q_4 = 2\beta / [g(1 + |B|)] - \sqrt{S^2 + W_2 / [g(1 + |B|)] + 4\beta^2 / [g^2(1 + |B|)^2]},
$$

$$
Q_5 = 2\beta / [g(1 - |B|)] - \sqrt{S^2 + W_2 / [g(1 - |B|)] + 4\beta^2 / [g^2(1 - |B|)^2]},
$$

$$
Q_6 = 2\beta / [g(1 + |B|)] - \sqrt{S^2 + W_2 / [g(1 + |B|)] + 4\beta^2 / [g^2(1 + |B|)^2]}.
$$
\[ Q_4 = 2\beta/[g(1 + |B|)] - \sqrt{S_0^2 + W_2/[g(1 + |B|)]} + 4\beta^2/[g^2(1 + |B|)^2]. \]  
(A.20)

Two solutions written above, (A.1) and (A.2), describe the trajectories inside two separatrix loops on the sphere (see Fig. 3a). The separatrix solution itself can be obtained in the limit \( k \to 0 \). On the separatrix trajectory, the value of the second integral is \( W_2 = -4gS_0^2(1 - |B|) - 4\beta/[g(1 - |B|)], \) \( C = (g/2)\sqrt{S_0^2g^2(1 + |B|)(1 - |B|)^2 - 4\beta^2(1 + |B|)} \), and the Stokes vector components for the loop embracing the fixed point \( \{0, 0, S_0\} \) have the form

\[
S_{1}^{(1)} = \pm \sqrt{\frac{1 - |B|}{2|B|}} \cosh[C\xi] \frac{G^2 - 16\beta^2B^2}{g(1 - B^2)(G\cosh[2C\xi] + 4\beta B)},
\]

\[
S_{2}^{(1)} = \pm \sqrt{\frac{1 + |B|}{2|B|}} \sinh[2C\xi] \frac{G\sqrt{G^2 - 16\beta^2B^2}}{g(1 - B^2)(G\cosh[2C\xi] + 4\beta B)},
\]

\[
S_{3}^{(1)} = 4\beta(1 - |B|) + \sqrt{2|B|(1 - |B|)[S_0^2g^2(1 - |B|^2) - 4\beta^2]}
\times \frac{4\beta B|\cosh[2C\xi] + \sqrt{2|B|(1 - |B|)[S_0^2g^2(1 - |B|^2) - 4\beta^2]}}{4\beta B + \sqrt{2|B|(1 - |B|)[S_0^2g^2(1 - |B|^2) - 4\beta^2] \cosh[2C\xi]}},
\]  
(A.21)

where \( G = [4\beta(1 - |B|)^2(\beta - S_0g(1 - B^2))] + g(1 - B^2)(S_0g(1 - B^2) - 4\beta)^{1/2} \). In this limit, the second solution gives the second separatrix loop embracing the opposite pole of the sphere, \( \{0, 0, -S_0\} \),

\[
S_{1}^{(2)} = \pm \sqrt{\frac{1 - |B|}{2|B|}} \cosh[C\xi] \frac{G^2 - 16\beta^2B^2}{g(1 - B^2)(G\cosh[2C\xi] - 4\beta B)},
\]

\[
S_{2}^{(2)} = \pm \sqrt{\frac{1 + |B|}{2|B|}} \sinh[2C\xi] \frac{G\sqrt{G^2 - 16\beta^2B^2}}{g(1 - B^2)(G\cosh[2C\xi] - 4\beta B)},
\]

\[
S_{3}^{(2)} = 4\beta(1 - |B|) - \sqrt{2|B|(1 - |B|)[S_0^2g^2(1 - |B|^2) - 4\beta^2]}
\times \frac{4\beta B - \sqrt{2|B|(1 - |B|)[S_0^2g^2(1 - |B|^2) - 4\beta^2] \cosh[2C\xi]}}{4\beta B \cosh[2C\xi] - \sqrt{2|B|(1 - |B|)[S_0^2g^2(1 - |B|^2) - 4\beta^2]}},
\]  
(A.22)

In the limit \( k \to 0 \), the solution \( S_{1}^{(1)} \) gives the fixed point on the upper pole of the Poincaré sphere,

\[
S_{1}^{(1)} = 0, \quad S_{2}^{(1)} = 0, \quad S_{3}^{(1)} = S_0,
\]  
(A.23)

for which the second integral takes the value \( W_2 = -4\beta|B|S_0 \). The solution \( S_{1}^{(2)} \), for the value of \( W_2 = 4\beta|B|S_0 \), gives another fixed point on the pole,

\[
S_{1}^{(2)} = 0, \quad S_{2}^{(2)} = 0, \quad S_{3}^{(2)} = -S_0.
\]  
(A.24)

The closed loops of trajectories beyond the separatrix are given by the other pair of solutions,

\[
S_{1}^{(3)} = \pm \sqrt{\frac{|B| - 1}{2|B|}} \text{sn}[C\xi, k] \frac{\sqrt{(Q_1 - Q_2)(Q_1 - Q_3)}}{Q_1 - Q_3 - (Q_1 - Q_2) \text{sn}^2[C\xi, k]},
\]

\[
S_{2}^{(3)} = \pm \sqrt{\frac{|B| + 1}{2|B|}} \text{cn}[C\xi, k] \frac{\sqrt{(Q_2 - Q_4)(Q_1 - Q_3) + (Q_4 - Q_3)(Q_1 - Q_2) \text{sn}^2[C\xi, k]}}{\sqrt{(Q_1 - Q_2)^{-1}(Q_1 - Q_3)^{-1}(Q_1 - Q_3 - (Q_1 - Q_2) \text{sn}^2[C\xi, k])}},
\]
\begin{equation}
S_3^{(3)} = \frac{Q_2(Q_1 - Q_3) - Q_3(Q_1 - Q_2)}{Q_1 - Q_3 - (Q_1 - Q_2) \, \text{sn}^2[C\xi, k]},
\end{equation}

and

\begin{align*}
S_1^{(4)} &= \pm \sqrt{\frac{|B| - 1}{2|B|}} \left( Q_2 - Q_3 \right) \, \text{sn}[C\xi, k] \frac{\sqrt{(Q_3 - Q_4)(Q_2 - Q_4)}}{Q_2 - Q_4 - (Q_3 - Q_4) \, \text{sn}^2[C\xi, k]}, \\
S_2^{(4)} &= \pm \sqrt{\frac{|B| + 1}{2|B|}} \, \text{cn}[C\xi, k] \frac{\sqrt{(Q_2 - Q_4)(Q_3 - Q_4) + (Q_3 - Q_4)(Q_2 - Q_4)}}{(Q_2 - Q_4)^{-1}(Q_3 - Q_4)^{-1}(Q_2 - Q_4 - (Q_3 - Q_4) \, \text{sn}^2[C\xi, k])}, \\
S_3^{(4)} &= \frac{Q_3(Q_2 - Q_4) - Q_2(Q_3 - Q_4)}{Q_2 - Q_4 - (Q_3 - Q_4) \, \text{sn}^2[C\xi, k]},
\end{align*}

where the parameters $C$ and $k$ are defined by expressions (A.16), (A.17). In the limit $k \to 1$, the solution $S_3^{(3)}$ gives a pair of fixed points corresponding to stationary elliptically polarized beams,

\begin{equation}
S_1^{(3)} = \pm \sqrt{S_0^2 - 4\beta^2/[g^2(1 - |B|)^2]}, \quad S_2^{(3)} = 0, \quad S_3^{(3)} = 2\beta/[g(1 - |B|)],
\end{equation}

for which the second integral takes the value $W_2 = -g(1 - |B|)S_0^2 - 4\beta^2/[g(1 - |B|)]$. The solution $S_2^{(4)}$ gives another pair of fixed points for which the second integral takes the value $W_2 = -g(1 + |B|)S_0^2 - 4\beta^2/[g(1 + |B|)]$,

\begin{equation}
S_1^{(4)} = 0, \quad S_2^{(4)} = \pm \sqrt{S_0^2 - 4\beta^2/[g^2(1 + |B|)^2]}, \quad S_3^{(4)} = 2\beta/[g(1 + |B|)].
\end{equation}

In the limit $k \to 0$, the trajectories (A.25), (A.26) degenerate into fixed points on the poles (A.27) and (A.28).

References