Interband resonant transitions in two-dimensional hexagonal lattices: Rabi oscillations, Zener tunnelling, and tunnelling of phase dislocations

Valery S. Shchesnovich\textsuperscript{1}, Anton S. Desyatnikov\textsuperscript{2}, and Yuri S. Kivshar\textsuperscript{2}

\textsuperscript{1} Centro de Ciências Naturais e Humanas, Universidade Federal do ABC, Santo André, SP, 09210-170 Brazil
\textsuperscript{2} Nonlinear Physics Centre, Research School of Physical Sciences and Engineering, Australian National University, Canberra ACT 0200, Australia

\texttt{vs\_shchesnovich@yahoo.co.uk}

Abstract: We study, analytically and numerically, the dynamics of interband transitions in two-dimensional hexagonal periodic photonic lattices. We develop an analytical approach employing the Bragg resonances of different types and derive the effective multi-level models of the Landau-Zener-Majorana type. For two-dimensional periodic potentials without a tilt, we demonstrate the possibility of the Rabi oscillations between the resonant Fourier amplitudes. In a biased lattice, i.e., for a two-dimensional periodic potential with an additional linear tilt, we identify three basic types of the interband transitions or Zener tunnelling. First, this is a quasi-one-dimensional tunnelling that involves only two Bloch bands and occurs when the Bloch index crosses the Bragg planes away from one of the high-symmetry points. In contrast, at the high-symmetry points (i.e., at the M and $\Gamma$ points), the Zener tunnelling is essentially two-dimensional, and it involves either three or six Bloch bands being described by the corresponding multi-level Landau-Zener-Majorana systems. We verify our analytical results by numerical simulations and observe an excellent agreement. Finally, we show that phase dislocations, or optical vortices, can tunnel between the spectral bands preserving their topological charge. Our theory describes the propagation of light beams in fabricated or optically-induced two-dimensional photonic lattices, but it can also be applied to the physics of cold atoms and Bose-Einstein condensates tunnelling in tilted two-dimensional optical potentials and other types of resonant wave propagation in periodic media.

@ 2008 Optical Society of America

OCIS codes: (050.0050) Diffraction and gratings; (050.5298) Photonic crystals; (050.4865) Optical vortices

References and links
1. Introduction

Electrons in crystalline solids or semiconductor superlattices, cold atoms in optical lattices, light beams in photonic crystals or waveguide arrays have the energies confined to the Bloch bands separated by band gaps. The system response to a weak linear potential (i.e., a weak constant tilt) is in the form of Bloch oscillations [1] and the interband transitions known as Zener tunnelling [2] (see also [3]). The case of an avoided crossing of two Bloch bands is described by a two-level model due to Zener [2], Landau [4] and Majorana [5]. Examples of Zener tunnelling include electrical breakdown in Zener diodes [6], electrical conduction in nanotubes [7] and super lattices [8], tunnelling of the Bose-Einstein condensates (BECs) in optical lattices [9, 10], and an optical analog of tunnelling in arrays of optical waveguides and photonic crystals [11, 12, 13].

The above studies, however, consider only the one-dimensional case. In the recent experiment [14], the interband transitions have been observed for the first time in a two-dimensional periodic structure of square symmetry (an optically-induced photonic lattice). Multi-dimensional optical lattices are also routinely used for trapping of ultracold atoms [15, 16] and BEC [17, 18] of degenerate quantum gases (see [19] and references therein), where more sophisticated lattice geometries for trapping of ultracold atoms have been shown experimentally, e.g., the Kagomé lattice [20], the dice lattice [21], and quasi-periodic lattices [22]. Simple theories are especially important for understanding the wave dynamics in the periodic structures. Recently, the theory of Zener tunnelling in the square lattices has been developed in [23], moreover, the three-fold Bragg resonance in the hexagonal lattices has been studied as well [24].

The purpose of this paper is to give a complete theory of the Zener tunnelling in the hexagonal periodic lattices and test it by extensive numerical simulations. Though the perturbation theory for band-gap structures we use is a subject of textbooks on the solid state physics (see, for instance, Ref. [25]), any detailed study of Zener tunnelling in band gap structures in more than one spatial dimension has never been attempted. In particular, only the case of avoided crossing of two Bloch bands, long known since the works of Zener, Landau and Majorana, is discussed in literature. We choose as the illustrative example the hexagonal lattice for the two reasons. First, the hexagonal lattice, as distinct from the square lattice, cannot be separable, i.e. it cannot be represented as a sum of two one-dimensional potentials. Second, the hexagonal periodic...
lattice is advantageous in the two dimensional case and most of the two-dimensional photonic crystals have the hexagonal symmetry.

Due to the experimental advances in fabrication of periodic structures for both light and matter waves the interest in wave dynamics in two-dimensional lattices is growing. For instance, numerical studies of Bloch oscillations and Zener tunnelling [26, 27] were recently performed. It was also shown that the nonlinearity of the governing equations, in the matter waves case, is responsible for many new features. Some of them can be studied already in the one-dimensional lattices: asymmetry of the tunnelling probabilities, found in Refs. [28, 29] and shown experimentally [9, 10], and modulational instability of Bloch waves [30] (see also Ref. [31]), also resulting in asymmetric nonlinear tunnelling. Moreover, a recent study of nonlinear tunnelling in the square two-dimensional lattices [32] shows the existence of intraband tunnelling (see also Ref. [33, 34] for the quantum case), absent in the linear multi-dimensional case and in the one-dimensional nonlinear case.

Our attention is concentrated on the linear wave dynamics in hexagonal lattices, the nonlinear case is relegated to a future study. This imposes an upper bound on the nonlinearity, which is discussed below (see section 2). We make a first significant step forward in understanding of the phenomenon of multi-dimensional Zener tunnelling by reducing the Schrödinger equation to a system of ordinary differential equations. It is shown, for instance, that the two-dimensional tunnelling in the hexagonal lattices is described by either the three- or six-level Landau-Zener-Majorana (LZM) model with various interesting reductions in the special cases. Moreover, we derive analytical expressions for some of the transition probabilities by using the recent results on the multi-level LZM models [35, 36, 37].

The paper is organized as follows. In section 2 we introduce the model equation describing tunnelling in tilted two-dimensional lattices and discuss the limits of its applicability to BEC tunnelling and photonic interband transitions. Then, we discuss the shallow-lattice approximation and show that it allows to reduce the most general hexagonal periodic lattice to a simpler form. In sections 3 and 4 we discuss, respectively, the Rabi oscillations and Zener tunnelling. We derive simple LZM-type models and obtain the analytical formulae for the tunnelling probabilities. In section 5 we consider the interband tunnelling of vortices. Throughout the paper, the analytical models are compared with the numerical simulations. Section 6 provides some concluding remarks.

### 2. Shallow-lattice approximation

We consider the model describing the propagation of paraxial optical beams in a planar periodic photonic structure, e.g., a two dimensional lattice optically induced in photorefractive crystal [14, 38],

\[
i\partial_t \Psi + \frac{1}{2} \nabla^2 \Psi + \varkappa \Delta n \Psi = 0, \quad (1)
\]

where \( \varkappa \) is the normalized nonlinear refractive index, \( \nabla^2 = \partial_x^2 + \partial_y^2 \) and \( \Delta n = \frac{I_e(x)+I_m(x)}{1+I_e(x)+I_m(x)} \), where \( x = (x, y) \), \( I_e(x) \) describes the optical lattice, and \( I_m(x) \) is a lattice tilt. We define the lattice potential as \( V(x) \equiv -\varkappa I_e(x) \) and assume the tilt to be linear, \( \varkappa I_m(x) = -\alpha x \). Equation (1) in the shallow lattice approximation, \( |V(x)| \ll 1 \) (the main approximation used below), and for a weak linear tilt, \( |\alpha|/2 \ll |V(d)/2 - V(0)| \), reduces to the linear Schrödinger equation

\[
i\partial_t \Psi = -\frac{1}{2} \nabla^2 \Psi + |V(x) + \alpha x| \Psi, \quad (2)
\]

with an inessential constant term (see [23] for more details). The dimensionless evolution variable \( t \) is the propagation distance in the case of an optical beam in a periodic photonic structure and the normalized time in the case of BEC. In the model, \( V(x) \) is the periodic lattice potential.
\[ V(x + d) = V(x), \] where \( d \) is one of the lattice periods, see Fig. 1(b), \( \alpha \) is the acceleration of the lattice in the case of BEC and the steepness of the refractive index tilt in the case of the optical beam propagation. Below, we denote by \( \mathbf{Q} \) a reciprocal lattice vector.

In the case of BEC, the Gross-Pitaevskii equation for the order parameter of BEC in a two-dimensional optical lattice (for further details consult [19]) can be also reduced to Eq. (2) if we allow for the following conditions. The parabolic trap along the transverse \( z \)-direction is assumed to be much stronger than the nonlinearity of BEC, so that the condensate performs the ground-state quantum motion along this direction, while the trap in the \((x, y)\)-plane is assumed to be weak and is neglected, i.e., it contains a large number of the lattice periods. The nonlinearity of BEC can be neglected if the Bloch oscillation period \( t_B \) is much less than the characteristic time \( t_{\text{non}} \) of the modulational instability development (see, for instance, [39]). \( t_B \ll t_{\text{non}} \sim 1/\gamma \), where \( \gamma \) is proportional to the \( s \)-wave scattering length multiplied by the number of BEC atoms per lattice cell [19]. Since \( t_B \sim |\mathbf{Q}| / |\alpha| \) with \( \mathbf{Q} = b_j \) (see Fig. 1(a)) and \( \alpha \) the component of the acceleration parallel to \( \mathbf{Q} \), we obtain the condition in the form \( \gamma \ll |\alpha| / |d| \) (since \( |b_j| \sim 1/|d| \)). Under the above conditions Eq. (2) describes the condensate tunneling in a titled two-dimensional optical lattice.

The shallow lattice approximation allows one to develop a fully analytical approach to the Zener interband transitions and reduce the governing equation to the multi-level LZM models. This approximation was already successfully used in [23, 24] and, moreover, showed an excellent qualitative agreement with the experimental results [14] for an arbitrary lattice. On the other hand, for a deep lattice, which is the other limiting case, the continuous Eq. (2) is replaced by a set of coupled discrete equations for the amplitudes of Wannier states, which is the so-called tight-binding approximation [26, 40, 41]. Clearly, the two limits are very different and assume different localization properties of the wave \( \Psi(x) \).

The interband transitions in a shallow lattice occur at the Bragg resonances, thus we assume that \( \Psi(x) \) has a narrow Fourier spectrum,

\[
\Psi(x, 0) = \int_{D} d\mathbf{k} C(\mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{x}},
\]

where we require that the size of \( D \) be much smaller than the size of the Brillouin zone. In other terms, the initial condition is broad enough and covers at least several lattice periods. This means using broad optical beams in photonic crystals (as it was done in the recent experiment [14]) and preparing the ultracold atoms or BEC in the state with a small momentum spread (as in the experiments of [9, 10]). In this case, \( C(\mathbf{k}) \) can be modelled by a sum of the Dirac delta-functions and the Zener tunnelling is described by an LZM model.

It is well known that at least three monochromatic plane waves \( E_i = E_i e^{i\mathbf{k}_i \cdot \mathbf{x} - i\omega} \) are necessary to produce a two-dimensional optical lattice, \( V = |\sum E_i|^2 \). To have a hexagonal lattice the wave vectors should satisfy the corresponding symmetry, i.e., they should transform under the rotation by \( \pi/3 \) as follows: \( \mathbf{k}_1 \rightarrow \mathbf{k}_2, \mathbf{k}_2 \rightarrow \mathbf{k}_3, \mathbf{k}_3 \rightarrow -\mathbf{k}_1 \). The general expression for a hexagonal lattice (not necessarily created by an interference) can be given as the following Fourier series, infinite in general,

\[
V = I_0 \left\{ \varepsilon_1 \sum_{i=1}^{3} \cos(\mathbf{b}_i \cdot \mathbf{x}) + \varepsilon_2 \sum_{i=1}^{3} \cos(2\mathbf{b}_i \cdot \mathbf{x}) + \varepsilon_3 \cos((\mathbf{b}_1 + \mathbf{b}_2) \cdot \mathbf{x}) + \cos((\mathbf{b}_2 + \mathbf{b}_3) \cdot \mathbf{x}) + \cos((\mathbf{b}_3 - \mathbf{b}_1) \cdot \mathbf{x}) + \ldots \right\},
\]

where the base vectors are \( \mathbf{b}_1 = \mathbf{k}_1 - \mathbf{k}_2, \mathbf{b}_2 = \mathbf{k}_1 - \mathbf{k}_3, \) and \( \mathbf{b}_3 = \mathbf{k}_2 - \mathbf{k}_3. \) Note that \( \mathbf{b}_3 = \mathbf{b}_2 - \mathbf{b}_1. \) On the r.h.s. of Eq. (4) we denote by the symbol “…” the omitted higher-order terms in the base.
In the absence of the external bias \( \mathbf{a} = 0 \) the interband transition can be observed in the form of periodic Rabi oscillations.  In the limit of shallow lattice, instead of expanding the solution over the Bloch waves, it is more convenient to use the Fourier version of the Schrödinger equation (2) directly. Setting \( \Psi(x,t) = \int d\mathbf{k} C(\mathbf{k},t) e^{i\mathbf{k} \cdot \mathbf{x}} \), expanding the lattice potential into the Fourier series,

\[
V(x) = \sum_\mathbf{Q} \hat{V}_Q e^{i\mathbf{Q} \cdot \mathbf{x}},
\]

3. Rabi oscillations

In the absence of the external bias \( \mathbf{a} = 0 \) the interband transition can be observed in the form of periodic Rabi oscillations. In the limit of shallow lattice, instead of expanding the solution over the Bloch waves, it is more convenient to use the Fourier version of the Schrödinger equation (2) directly. Setting \( \Psi(x,t) = \int d\mathbf{k} C(\mathbf{k},t) e^{i\mathbf{k} \cdot \mathbf{x}} \), expanding the lattice potential into the Fourier series,

\[
V(x) = \sum_\mathbf{Q} \hat{V}_Q e^{i\mathbf{Q} \cdot \mathbf{x}},
\]
obtains the equation for the Fourier coefficients

\[ i\partial_t C(q - Q) = E_0(q - Q)C(q - Q) + \sum_{Q'} \hat{V}_{Q'Q} C(q - Q'), \]

(7)

where \( E_0(q) = \frac{1}{2}q^2 \). The usual resonant perturbation theory can be used to determine the Bloch bands [25]. In essence, the band gaps appear at the resonant Bragg planes defined as points \( q \) where \( E_0(q - Q) = E_0(q') \). Most of the Fourier amplitudes are nonresonant and can be neglected in the first-order approximation. Then Eq. (7) predicts oscillations between the resonant peaks in Fourier space defined by Bragg condition \( E_0(q - Q) = E_0(q - Q') \) [42]. To observe these oscillations the input wave (or beam) must be placed on the boundary of the first Brillouin zone. In the simplest case, i.e., away from the high-symmetry points, there are just two equivalent points on the boundary (\( Q/2 \) and \( -Q/2 \) below) and the oscillations are equivalent to those in the two-level system:

\[ i\partial_t C(\delta q + Q/2) = E_0(\delta q + Q/2)C(\delta q + Q/2) + \hat{V}_{Q} C(\delta q - Q/2), \]

(8)

\[ i\partial_t C(\delta q - Q/2) = E_0(\delta q - Q/2)C(\delta q - Q/2) + \hat{V}_{Q} C(\delta q + Q/2), \]

(9)

where we have used \( \hat{V}_{Q} = \hat{V}_{Q} \) and introduced the running index \( \delta q \) for the points of the two resonant Fourier peaks. Since the Bloch waves are approximated as linear combinations of plane waves, the system (8)-(9) corresponds to the inter band oscillations between two Bloch bands. By analogy, these oscillations can be called Rabi oscillations. Rabi oscillations between several Bloch bands can be realized by placing the wave on one of the high-symmetry points on the boundary of the first Brillouin zone.

Figure 2 and the movie Rabi.avi demonstrate the results of the simulations of Eq. (2) with a Gaussian beam as the initial condition, namely we use \( \exp(-x^2/2w^2 + iq_0x) \) with \( w = 20 \) and \( q_0 = b_1/2 \), i.e., in Fourier space the beam is initially at the right X-point, see Fig. 1(a). We use split-step beam propagation method and monitor the dynamics also in the Fourier domain, see Fig. 2(b). The energy is periodically transferred between two X-points (\( \pm b_1/2 \)) and, for a quantitative comparison with the predictions of the LZM system (8–9), we integrate the intensities of two interacting beams in the Fourier domain to obtain their normalized powers \( P_{1,2} \).

The results presented in Fig. 2(c) allow to estimate, roughly, the period of these oscillations to be \( \sim 28.8 \). At the same time, the system (8–9) has a solution \( (\delta q = 0) \) in terms of harmonic functions with the period \( T = \pi/\hat{V}_{Q} = \pi/(\epsilon_1 I_0) \approx 41.9 \), for the used parameters of the lattice \( I_0 = 0.1 \) and \( \epsilon_1 = 3/2 \). Most probably, the disagreement is due to the transitional dynamics in Fig. 2 and averaging over longer propagation time provides better comparison with analytical predictions, similar as in the square lattice case [42].
In general, Rabi oscillations also show up in the real space as the oscillations of the average position of the beam. Besides performing oscillations, however, the beam propagates in the lattice due to the non-zero width of its Fourier image. Indeed, for $\delta q \neq 0$ there is the frequency mismatch between the corresponding Fourier amplitudes: $E_0(\mathbf{Q}/2 + \delta \mathbf{q}) - E_0(-\mathbf{Q}/2 + \delta \mathbf{q}) = \mathbf{Q}\delta \mathbf{q}$ resulting in a higher average Fourier power of the input wave over the Bragg reflected one (see also [42]). In our simulations in Fig. 2 we choose relatively broad initial beam so that the oscillations and drift of its position are minimized.

4. Zener tunnelling

Since we consider the states (waves packets or beams) $\Psi'(x)$ with a narrow Fourier spectrum, we can use the expansion over the Bloch waves with a definite $t$-dependent Bloch index $\mathbf{q} = \mathbf{q}(t)$, similar to Houston’s approach [43] in the theory of accelerating electrons. In particular, we represent the solution to Eq. (2) in the following convenient form

$$\Psi'(x,t) = \int_D d\mathbf{k} C(k,t)e^{i(k-\mathbf{q})x}.$$ (10)

This representation, derived from the plane wave solution in a linear potential $\Psi'(x,t) = \exp\{i[k_0 - \mathbf{q}](x - t\int d\tau \mathbf{k}_0 - \mathbf{q} \tau)^2/2\}$, is an alternative to having a partial derivative in the governing equation in Fourier space, which would account for the linear potential according to the rule $x \rightarrow i\delta k$. Indeed, we get from Eq. (10)

$$i\partial_t C(\mathbf{q} - \mathbf{Q}) = E_0(\mathbf{q} - \mathbf{Q} - \mathbf{q} \tau)C(\mathbf{q} - \mathbf{Q}) + \sum_{\mathbf{Q}'} \tilde{V}_{\mathbf{Q}'} C(\mathbf{q} - \mathbf{Q}').$$ (11)

By switching to the “interaction picture” of the perturbation theory (where the lattice is considered a perturbation) by the transformation $C(\mathbf{q} - \mathbf{Q},t) = e^{-i\int_0^t d\tau E_0(\mathbf{q} - \mathbf{Q} - \mathbf{q} \tau)}C(\mathbf{q} - \mathbf{Q},t)$ one sees that any two coefficients $C(\mathbf{q} - \mathbf{Q})$ and $C(\mathbf{q} - \mathbf{Q}')$ are effectively coupled on the time interval where the property $E_0(\mathbf{q} - \mathbf{Q} - \mathbf{q} \tau) \approx E_0(\mathbf{q} - \mathbf{Q} - \mathbf{q} \tau)$ is satisfied, otherwise the coupling coefficients are oscillating about zero. Thus the interband transitions take place on the Bragg resonance planes. It is convenient to explicitly account for the Bragg resonance by defining the resonant point $\mathbf{q}_{\text{res}}$ by setting $\mathbf{q} - \mathbf{Q} - \mathbf{q} \tau = \mathbf{q}_{\text{res}} - \mathbf{Q} - \mathbf{q} \tau(t - t_0)$ in the energy $E_0$. For simplicity, below we set $t_0 = 0$ (in this case one obtains the governing LZM-type models in the standard form). There are three types of Zener interband transitions in the hexagonal lattices: (i) the quasi one-dimensional tunnelling (section 4.1), (ii) tunnelling between three Bloch bands at the $\Gamma$-point (section 4.2), and (iii) between the six Bloch bands at the $\Lambda$-point (section 4.3).

4.1. Quasi-one-dimensional Zener transitions

The quasi one-dimensional Bragg resonance takes place when the Bloch index crosses the Bragg planes outside the small neighborhoods of the high-symmetry points $\Gamma$ and $\Lambda$ (of radius $R \sim L_0$). This is a transition between two Bloch bands at an avoided crossing, which takes place along one of the borders of the irreducible Brillouin zone, i.e. the triangle $\Gamma\Lambda\Lambda$ in Fig. 1, to which the Bragg plane is equivalent (after performing rotations by multiples of $\pi/3$ and translations by the reciprocal lattice vectors). Let us briefly recall the derivation, which is similar as in the case of the square lattice [23] (see also [28, 29, 44] for the one-dimensional case). By keeping only the resonant terms in the potential and in the Bloch wave, we arrive at the following expressions:

$$V_j = \frac{\epsilon J_0}{2} \left(e^{i\mathbf{Q}_j x} + e^{-i\mathbf{Q}_j x}\right), \quad \Psi_j = C_1(t)e^{i\mathbf{q}(t)\mathbf{x}} + C_2(t)e^{i\mathbf{q}(t) - \mathbf{Q}_j\mathbf{x}}.$$ (12)
The Bragg resonance point is $q_j = Q_j/2$. Substituting the expression for the Bloch wave into Eq. (2) and requiring cancellation of the terms linear in $x$, which gives $\mathbf{q} = -\mathbf{a}$ (the dot denotes derivative with respect to $t$), we get a system of coupled equations for the incident $C_1$ and Bragg reflected $C_2$ amplitudes of the Bloch wave. By setting $q = Q_j/2 - \alpha t$ and $(C_1, C_2) = e^{i\theta(t)}(c_1, c_2)$, with $\phi = (q^2(t) + |q(t) - Q_j|^2)/4$, the latter system is cast in the form of Zener [2], Landau [4], and Majorana [5]:

$$i\dot{c}_1 = -\frac{(Q_j \alpha t)}{2} c_1 + \frac{e_j I_0}{2} c_2, \quad i\dot{c}_2 = \frac{(Q_j \alpha t)}{2} c_2 + \frac{e_j I_0}{2} c_1.$$  \(13\)

System (13) is Hamiltonian with the adiabatic energy levels corresponding to the two Bloch bands $E_j$ at vanishing crossing: $E_{1,2} = \pm \sqrt{q^2 + (e_j I_0/2)^2}$, where $q = (Q_j \alpha t)/2$ is the running band parameter. All quasi one-dimensional cases are treated in the same way: there is a single resonant reciprocal lattice vector $Q$ which defines a Bragg plane and the probability of tunneling is given by the LZM formula:

$$P = \exp \left\{ -\frac{\pi |\mathbf{V}|^2}{Q|\alpha|} \right\},$$  \(14\)

with $\alpha_\perp$ being the component of the bias perpendicular to the respective Bragg plane. Thus, Eq. (14) gives the probability of transition between the two Bloch bands $E_{1,2}(q)$, defined for the $j$-th band as $P_j = |c_j(\infty)|^2$ for the initial condition $|c_j(-\infty)| = 1$. Due to the symmetry $t \rightarrow -t$ and $c_1 \rightarrow c_2^\ast, c_2 \rightarrow c_1^\ast$ the two probabilities coincide. Note also that, in the case of a sufficiently small bias $|\alpha| \ll (\epsilon_i I_0)^2$, Zener tunneling is negligible when crossing the border of the first Brillouin zone (away from the high-symmetry point M). This means that the wave (beam) is totally Bragg-reflected at the boundary of the first Brillouin zone, i.e., it performs Bloch oscillations. There are three distinct cases corresponding to the three inequivalent Bragg planes, each one being equivalent to one side of the irreducible Brillouin zone. Recall that in the derivation of the LZM model (13) we have subtracted the mean value of the Bloch energy $\bar{E} = (Q_j)^2/8$ (by using the phase transformation) which specifies the two bands involved in the tunneling transition.

I. Zener transitions on the XM-border. In this case the resonant reciprocal lattice vector is $Q = b_1$ (equivalently $b_2$ or $b_3$). The resonant Fourier components of the lattice are those with the coefficient $\epsilon_1$ in formula (4) and the LZM formula (14) becomes $P = \exp \left\{ -\frac{\pi \epsilon_1^2 b_{10}^2}{4|\alpha_1|} \right\}$. This is the transition between the first and the second Bloch bands (see Fig. 1(d)), in fact, it occurs when the Fourier index crosses the border of the first Brillouin zone away from the M point.

II. Zener transitions on the GM-border. The resonant reciprocal lattice vector is $Q = 2b_1$ (or its equivalent). The resonant Fourier components of the lattice are those with the coefficient $\epsilon_2$. The shift by $-b_1$ and rotation by $-\pi/3$ transforms the resonance point to a point lying on the GM-border. The probability of tunnelling reads $P = \exp \left\{ -\frac{\pi \epsilon_2^2 b_{10}^2}{4\sqrt{3} |\alpha_1|} \right\}$. This is the transition between the second and the third Bloch bands along the GM-border, in contrast to the above transition, it occurs outside the first Brillouin zone in the extended zone picture.

III. Zener transitions on the TX-border. The resonant reciprocal lattice vector is equivalent to $Q = b_1 + b_2$ and the resonant Fourier components are those with the coefficient $\epsilon_3$. The rotation by $\pi/3$ followed by the shift by $-b_3$ transforms the resonance point to a point lying on the TX-border. We obtain the probability as follows $P = \exp \left\{ -\frac{\pi \epsilon_3^2 b_{10}^2}{4\sqrt{3} |\alpha_1|} \right\}$. This is the transition between the third and the fourth Bloch bands along the TX-border and it also occurs outside the first Brillouin zone in the extended zone picture.

(C) 2008 OSA 1 September 2008 / Vol. 16, No. 18 / OPTICS EXPRESS 14084
Fig. 3. Quasi-one-dimensional tunnelling through the X-point with the tilt directed along \( \mathbf{b}_1 \), see also movie 1D.avi. (a,b) Intensity in the real (top) and Fourier (bottom) spaces are shown for \( t = 0 \) in (a) and \( t = 10 \) in (b), corresponding dynamics of beam powers is shown in (d, solid lines). Solutions to the LZM system Eq. (13) are shown in (c) and (d, dashed lines). See text for the details and parameter values.

To illustrate the dynamics of the one-dimensional Zener tunnelling we choose the simplest case of the tunnelling through the X-point (case I. above), similar to Rabi oscillations in Fig. 2. We show in Fig. 3 and the movie 1D.avi results of the simulations with the parameters \( |\mathbf{b}_1| = 2 \), \( I_0 = 0.1 \), \( |\alpha| = 0.05 \), and \( \epsilon_1 = 3/2 \). The LZM result here is \( P = \exp(-9\pi/80) \approx 0.7 \) and it is shown with horizontal dashed line in Fig. 2(c), together with the solution to Eq. (13) with initial condition \( c_1(-\infty) = 1 \) and \( c_2(-\infty) = 0 \). The numerical simulations with a finite beam in Fig. 3(d, solid lines) reproduce the qualitative features of the asymptotic LZM result in Fig. 3(c). To have a quantitative comparison with the numerics one has to match the initial conditions. Instead of the asymptotic initial condition at minus infinity, a better account is given by the LZM system Eq. (13) with the initial condition \( t_0 = (q_N - q_0)/|\alpha| \), i.e. \( c_1(t_0) = 1 \) and \( c_2(t_0) = 0 \). By doing so, we were able to find an excellent agreement between direct numerical results (solid lines in Fig. 2(d)) and the solution to LZM system (dashed lines), including the value of the tunnelling efficiency \( \sim 0.77 \).

4.2. Zener transitions between three Bloch bands

Bragg resonance at the M-point is described by the three-fold resonance. Indeed, from Fig. 1 one can see that an M-point of the hexagonal lattice has exactly two equivalent points, \( M' \) and \( M'' \), such that the translations between all these points are given by the reciprocal lattice vectors \( \mathbf{b}_j \). To derive the corresponding LZM model we note that the resonant terms in the lattice potential and in the Bloch wave can be cast as follows:

\[
V_{\text{res}} = \frac{\epsilon_1 I_0}{2} \left[ e^{i(q_M + q_{M''} - q_M)x} + e^{i(q_M - q_{M''} + q_M)x} + e^{i(q_{M''} - q_M)x} + \text{c.c.} \right],
\]

\[
\Psi = C_1(t) e^{i(q_1)x} + C_2(t) e^{i(q_1 + q_{M''} - q_M)x} + C_3(t) e^{i(q_1 + q_{M''} - q_M)x},
\]

where \( q_M, q_{M'}, \) and \( q_{M''} \) are the vectors connecting the \( \Gamma \)-point with the corresponding M-point. The Fourier amplitudes of the Bloch wave are related to the M-points as follows: \( C_1 \rightarrow M, \)
$C_2 \rightarrow M'$, and $C_3 \rightarrow M''$. Setting $C_j = e^{i\phi(t)}c_j$, with $\phi = (q_M^2 + \alpha^2 t^2)/2$, and proceeding as in the derivation of the system (13) we obtain the three-level system

\begin{align*}
  i\dot{c}_1 &= -(\nu_1 t)c_1 + \frac{\epsilon_1 l_0}{2}(c_1 + c_3), \quad (17) \\
  i\dot{c}_2 &= -(\nu_2 t)c_2 + \frac{\epsilon_1 l_0}{2}(c_1 + c_3), \quad (18) \\
  i\dot{c}_3 &= -(\nu_3 t)c_3 + \frac{\epsilon_1 l_0}{2}(c_1 + c_2), \quad (19)
\end{align*}

where $\nu_1 = q_M \alpha$, $\nu_2 = q_M \alpha$, and $\nu_3 = q_M \alpha$. The coefficients $\nu_j$ satisfy the obvious constraint $\nu_1 + \nu_2 + \nu_3 = 0$. The invariance of the system (17)-(19) with respect to the $\pi/3$-rotation is evident from the corresponding transformation $\nu_1 \rightarrow \nu_2 \rightarrow \nu_3 \rightarrow \nu_1$. The coefficients $\nu_j$ can be cast as follows: $\nu_1 = (\ell/\alpha)|\cos \theta|$, $\nu_2 = -\ell (|\alpha| (\cos \theta - \sqrt{3} \sin \theta)/2$, and $\nu_3 = -\ell (|\alpha| (\cos \theta + \sqrt{3} \sin \theta)/2$, where $\ell = |q_M| = b/\sqrt{3}$ with $b = |b|$ (in our case $b = 2$) is the length of the hexagon side in Fig. 1 and $\theta$ is the polar angle of the bias direction counting from the $\Gamma M$-line (the choice of a particular $\Gamma M$-line just sets the ordering of the Fourier coefficients $c_j$). Since the transformation $\theta \rightarrow -\theta$, i.e., $c_2 \rightarrow c_3$ and $c_3 \rightarrow c_2$, leaves the system (17)-(19) invariant, the polar angle can be restricted to $0 \leq \theta \leq \pi/3$.

The first three Bloch bands along the direction of the tilt in the vicinity of the M-point (see also Fig. (1d)) are given by the set of the adiabatic energy levels of the system (17)-(19). Though the analytical result for the band structure in the vicinity of the M-point is quite complicated, at the M-point itself the values are given by (compare with Fig. 1(d))

\[ E_{\text{res}}(M) = \frac{b^2}{6} + \left[ -\frac{\epsilon_1 l_0}{2}; -\frac{\epsilon_1 l_0}{2}; -\frac{\epsilon_1 l_0}{2} \right], \quad (20) \]

where we have used that $|q_M| = b/\sqrt{3}$. Hence, in the case of the hexagonal lattice (4), two of the three Bloch bands collide at the M-point (in the case of the lattice (5) the first two bands collide, while in the case of the “triangular” lattice the second and third bands collide at the M-point). It is not known whether the $n$-level LZM system admits an analytical solution for all transition probabilities in the general case (see [35, 36, 37] for further discussion). However, the probabilities of two transitions are known in the analytical form. These are the transitions $m \rightarrow m$, where $m$ is the diagonal index of the minimal or maximal value of the coefficient at $t$ in the system (which must be unique). The probabilities are given by the general formula [35, 37] \( P_{m \rightarrow m} \equiv |c_m(\infty)|^2 \) for \( |c_m(-\infty)| = 1 \)

\[ P_{m \rightarrow m} = \exp \left\{ -2\pi \sum_{j \neq m} \frac{|\Delta_{mj}|^2}{|V_m - V_j|} \right\}, \quad (21) \]

where $V_j$ is the coefficient at $t$ in the $j$-th equation in the system and $\Delta_{mj}$ is the coupling coefficient between the $j$-th and $m$-th equations. In the generic case, i.e., when $0 < \theta < \pi/3$, we have $\nu_1 > \nu_2 > \nu_3$ in Eqs. (17)-(19) and the transition probabilities $P_{1 \rightarrow 1}$ and $P_{3 \rightarrow 3}$ are

\[ P_{1 \rightarrow 1} = \exp \left\{ -\frac{\pi \epsilon_1^2 l_0^2}{2} \left[ \frac{1}{\nu_1 - \nu_2} + \frac{1}{\nu_1 - \nu_3} \right] \right\}, \quad (22) \]

\[ P_{3 \rightarrow 3} = \exp \left\{ -\frac{\pi \epsilon_1^2 l_0^2}{2} \left[ \frac{1}{\nu_1 - \nu_3} + \frac{1}{\nu_2 - \nu_3} \right] \right\}. \quad (23) \]

Below we consider in more details two special cases, $\theta = 0$ and $\theta = \pi/3$, when Eqs. (17)-(19) can be reduced to the LZM system for two amplitudes.
I. Special case $\theta = 0$: symmetric three-fold Zener transitions.

In this case (the tilt is along the $\Gamma$M-line) we have $v = -v_2 = -v_3 = v_1 / 2$ with $v = |\alpha| / 2$ and it is convenient to introduce new amplitudes related to the original Bloch amplitudes by orthogonal transformation $b_1 = (c_2 - c_1) / \sqrt{2}$, $b_2 = (c_2 + c_3) / \sqrt{2}$, and $b_3 = c_1$. The amplitude $b_1$ decouples from the system, while $b_2$ and $b_3$ remain coupled:

\[
\begin{align*}
ib_1 &= \left( vt - \frac{\epsilon_1 I_0}{2} \right) b_1, \\
ib_2 &= \left( vt + \frac{\epsilon_1 I_0}{2} \right) b_2 + \frac{\epsilon_1 I_0}{\sqrt{2}} b_3, \\
ib_3 &= -2vt b_3 + \frac{\epsilon_1 I_0}{\sqrt{2}} b_2.
\end{align*}
\]

Equation (24) is readily solved, $b_1(t) = b_1(0) \exp \left( -ivt^2 / 2 + i\epsilon_1 I_0 t / 2 \right)$, and the system (25)-(26) has an additional integral of motion, $|b_2|^2 + |b_3|^2 \equiv C = \text{const}$. We employ the initial conditions $c_1(-\infty) = 1$ and $c_{2,3}(-\infty) = 0$, so that $C = 1$. It follows from the initial condition that $b_1(t) = 0$ and, hence, $c_2 = c_3 = b_2 / \sqrt{2}$ for all $t$. The conservation of norm can be expressed now as $|c_{2,3}|^2 = (1 - |c_1|^2) / 2$ and we refer to this case as the symmetric tunnelling. It was analyzed numerically in [24] and here we would like to stress several additional details.

Along the $\Gamma$M-line the Bloch bands have the structure about the M-point shown in Fig. 4(a), where $E_1 = (v_1 t - \epsilon_1 I_0) / 2$, $E_{2,3} = (-v_1 t + \epsilon_1 I_0 + 3\sqrt{D(t)}) / 4$, with $D(t) = \nu_1 t^2 - \epsilon_1^2 I_0^2 + 2\epsilon_1 \nu_1 t / 3$ ($D(t) > 0$ for $\epsilon_1 I_0 < 3$, i.e., for a shallow lattice). The tunnelling probability between the Bloch bands corresponding to the system (25)-(26) is still given by the probability $P_{1 \rightarrow 0}$ of Eq. (22), which in this case becomes the LJM result $P = \exp \left\{ -\frac{2\pi \nu_1 I_0^2}{3
u_1} \right\}$. An example of tunnelling through the M-point is given in Fig. 4(b,c), see also [24] and the movie 3D0.avi. It is interesting to note that the three-beam interference in Fig. 4(c) reproduces the wave used to optically induce the lattice itself [45], together with its regular phase structure shown in Fig. 4(d). The honeycomb pattern of phase dislocations, or optical vortices [46], is the signature of the corresponding Bloch wave [47] at the M-point of hexagonal lattice.

We conclude that Zener tunnelling can be employed for the excitation of specific Bloch waves and thus characterization of photonic structures. In the nonlinear regime optical vortices can form localized states, or vortex solitons [48], and novel multi-vortex states bifurcate from the vortex lattice [47]. Similar effects should be expected to occur with BEC and can be of interest for the excitation of vortex lattices.
II. Special case $\theta = \pi/3$: Zener tunnelling of Rabi oscillations. We have in this case $\nu_1 = \nu_2 = -\nu_3/2 = -\nu$ (note that now $\nu = -l|\alpha|/2$). Introducing the amplitudes $b_1 = (c_1 - c_2)/\sqrt{2}$, $b_2 = (c_1 + c_2)/\sqrt{2}$, and $b_3 = c_3$ we obtain formally the same system (24)-(26) (in fact, complex conjugate to it with inverted time $t \to -t$, if one takes into account change of sign of $\nu$). The initial conditions now read $b_1(0) = b_2(0) = 1/\sqrt{2}$ and $b_3(0) = 0$, so that $|b_2|^2 + |b_3|^2 = 1/2$. It follows from the solution to Eq. (24) that $c_1(t) = c_2(t) + \exp \left(-i\nu t^2/2 + i\epsilon_1 I_0 t/2\right)$.

Although now the reduction of essentially asymmetric three-fold resonance to the two-level LZM system is not so obvious as in previous case, its meaning becomes clear from the solution presented in Fig. 5(a). The Fourier components $c_1$ and $c_2$ form a Rabi oscillating state, described by the two-level system (17)-(18) (equivalent up to a phase transformation to the system (8)-(9) with $Q = b_1/2$ and $\tilde{V}_Q = \epsilon_1 I_0/2$) before and after the three-fold resonance crossing at $t \sim 0$, where they couple to the third Fourier amplitude $c_3$ (away from the resonance the coupling is non-resonant and can be neglected, i.e., $c_3$ rapidly varies with respect to $c_1$ and $c_2$). In the equivalent Bloch band population interpretation the Rabi oscillations are between the first and the second Bloch bands in Fig. 1(d) (along the XM-line) before the resonance crossing and between the second and third bands (now along the GM-line) after the crossing. Before the resonance, the first two band population amplitudes are, in fact, given by $b_1$ and $b_2$ (and the third by $b_3 = c_3$). After the resonance crossing $b_1$ and $b_2$ now give the second and third band populations, while $b_3$ is the population of the first band. As the result of the resonance, a part of the Rabi energy (given by $|b_3|^2$) is left in the first band and the amplitude of oscillations is reduced (recall the conservation law $|b_1|^2 = 1/2 - |b_3|^2$). Thus the tunnelling of the oscillating state, described by the LZM system (25)-(26), can be thought of as Zener tunnelling of Rabi oscillations.
with the onset of oscillations between LZM system in Fig. 5(e). Note that after the tunnelling starts to develop, the LZM system shows Rabi oscillations, we could successfully match these results with corresponding solution to the short time which our main moving beam spends near M-point is not sufficient to observe size, while the tunneled beam powers are plotted in Fig. 5(e), see also movie {±cal lattice vectors

In general, the Bragg resonance at the 4.3. Zener transitions between six Bloch bands

In general, the Bragg resonance at the Γ-point is six-fold due to the fact that each of the reciprocal lattice vectors \{±\textbf{b}_1,±\textbf{b}_2,±\textbf{b}_3\} if taken from the center of the Brillouin zone would point at one of the six nearest equivalent Γ-points, see Fig. 6(a), and the Bragg vectors giving the translations between these Γ-points correspond to non-zero Fourier amplitudes of the lattice (4) if all \(\varepsilon_j \neq 0\).

Let us assume that the resonance is at \(t = 0\) and the Bloch index is \(\textbf{q}(0) = \textbf{q}_{\Gamma_1} = \Gamma_1\), i.e., \(\textbf{q} = \textbf{q}_{\Gamma_1} - \alpha \textbf{t}\). Then the corresponding resonant terms of the Bloch wave can be cast as follows

\[
\Psi = C_1(t)e^{i\textbf{q}(0)\textbf{t}} + \sum_{l=2}^{6} C_l(t)e^{i\textbf{q}(l)\textbf{t} - \textbf{q}_{\Gamma_1}\textbf{t}}, \tag{27}
\]

where \(\textbf{q}_{\Gamma_1}\) is the reciprocal lattice vector connecting the center of the Brillouin zone with the \(\Gamma_1\)-point. Setting \(C_j = e^{i\phi_j(t)}c_j\), with \(\phi = (q_{\Gamma_1}^2 + \alpha^2 t^2)/2\), and following the same procedure as above we get the system of equations for the Fourier amplitudes \(c_j\) of the Bloch wave. Defining \(\mathbf{C} = (c_1,...,c_6)^T\) we have

\[
i\mathbf{C} = \left(\mathbf{\Lambda} + \frac{\mathbf{I}_0}{2}\mathbf{H}\right)\mathbf{C}, \quad \mathbf{H} = \begin{pmatrix} A & B \\ B & A \end{pmatrix}, \tag{28}
\]

with \(\mathbf{A} = \text{diag}(-\lambda_1, -\lambda_2, -\lambda_3, \lambda_1, \lambda_2, \lambda_3)\), where \(\lambda_j = b_j \alpha, j = 1,2,3\), and

\[
\mathbf{A} = \begin{pmatrix} 0 & \varepsilon_1 & \varepsilon_3 \\ \varepsilon_1 & 0 & \varepsilon_3 \\ \varepsilon_3 & \varepsilon_1 & 0 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} \varepsilon_2 & \varepsilon_3 & \varepsilon_1 \\ \varepsilon_3 & \varepsilon_2 & \varepsilon_1 \\ \varepsilon_1 & \varepsilon_3 & \varepsilon_2 \end{pmatrix}.
\]

The matrix \(\mathbf{H}\) is a circulant, i.e., it is a Toeplitz matrix, \(H_{i,j} = h(i - j)\), with the property \(h(l \pm 6) = h(l)\). This, together with the transformation \(\lambda_1 \rightarrow \lambda_2 \rightarrow \lambda_3 \rightarrow -\lambda_1\) under the rotation...
by $\pi/3$, guarantees the invariance of the system (28) under the rotations multiple of $\pi/3$. Due to $b_3 = b_2 - b_1$, the coefficients $\lambda_j$ satisfy the identity $\lambda_3 = \lambda_2 - \lambda_1$. They can be given as follows: $\lambda_1 = b|\alpha|\cos \varphi$, $\lambda_2 = b|\alpha|\cos \varphi + \sqrt{3}\sin \varphi)/2$, and $\lambda_3 = b|\alpha|(-\cos \varphi + \sqrt{3}\sin \varphi)/2$, with $\varphi$ being the polar angle of the bias direction counting from the $\Gamma\Gamma_1$-line. The angle $\varphi$ can be restricted to $0 \leq \varphi < \pi/3$ by the rotation symmetry.

The Bloch band structure about the $\Gamma$-point corresponding to the system (28), i.e., the six resonant Bloch bands, cannot be obtained in the analytical form. However, at the $\Gamma$-point itself the six resonant Bloch bands assume the following values

$$E_{\text{res}}(\Gamma) = \frac{b^2}{2} + \frac{l_0}{2}[\Delta_1; \Delta_2; \Delta_3; \Delta_4],$$

with $\Delta_1 = \varepsilon_2 - \varepsilon_1 - \varepsilon_3$, $\Delta_2 = \varepsilon_1 - \varepsilon_2 - \varepsilon_3$, $\Delta_3 = 2(\varepsilon_3 - \varepsilon_1) - \varepsilon_2$, and $\Delta_4 = 2(\varepsilon_3 + \varepsilon_1) + \varepsilon_2$. For the lattice (5) we get (we use $b = 2$, compare with Fig. 1(d))

$$E = 2 + \begin{bmatrix} \frac{-9l_0}{8}; \frac{-7l_0}{8}; \frac{-7l_0}{8}; \frac{3l_0}{8}; \frac{3l_0}{8}; \frac{17l_0}{8} \end{bmatrix},$$

while for the “triangular” lattice we have

$$E = 2 + \begin{bmatrix} \frac{-3l_0}{8}; \frac{-3l_0}{8}; \frac{-l_0}{8}; \frac{-l_0}{8}; \frac{-l_0}{8}; \frac{9l_0}{8} \end{bmatrix}.$$ 

In the case when there is a single minimal or maximal value of $\pm \lambda_j$ two tunnelling probabilities corresponding to the system (28) can be given in the analytic form by using formula (21). Noticing that $\lambda_1 > \lambda_3$ and $\lambda_2 > -\lambda_3$ (the equality sign for $\varphi = 0$) we have the two possible orderings of the coefficients:

$$\lambda_1 > \lambda_2 \geq -\lambda_3 > \lambda_3 \geq -\lambda_2 = -\lambda_1 \text{ for } 0 \leq \varphi < \pi/6,$$

$$\lambda_2 > \lambda_1 > \lambda_3 \geq -\lambda_3 > -\lambda_1 \geq -\lambda_2 \text{ for } \pi/6 < \varphi < \pi/3.$$

Thus, from the general result Eq. (21) we obtain:

$$P_{1\rightarrow 1} = P_{4\rightarrow 4} = \exp \left\{ -\frac{\pi l_0}{2} \left[ \frac{\varepsilon_1^2}{\lambda_1 - \lambda_2} + \frac{\varepsilon_1^2}{\lambda_1 + \lambda_3} + \frac{\varepsilon_3^2}{\lambda_1 - \lambda_3} + \frac{\varepsilon_3^2}{\lambda_1 + \lambda_2} + \frac{\varepsilon_2^2}{2\lambda_1} \right] \right\},$$

for $0 < \varphi < \pi/6$ and

$$P_{2\rightarrow 2} = P_{5\rightarrow 5} = \exp \left\{ -\frac{\pi l_0}{2} \left[ \frac{\varepsilon_1^2}{\lambda_2 - \lambda_1} + \frac{\varepsilon_1^2}{\lambda_2 + \lambda_3} + \frac{\varepsilon_3^2}{\lambda_2 - \lambda_3} + \frac{\varepsilon_3^2}{\lambda_2 + \lambda_2} + \frac{\varepsilon_2^2}{2\lambda_2} \right] \right\}.$$ 

for $\pi/6 < \varphi < \pi/3$. Under the rotation by $\pi/3$, due to the transformation $\lambda_1 \rightarrow \lambda_2 \rightarrow \lambda_3 \rightarrow -\lambda_1$, the probabilities in Eq. (30) go into those of Eq. (31), as it should be.

The most interesting cases of the six-fold resonance are for the special (i.e., symmetric in the lattice) bias directions, when the six-level system (28) decouples into two subsystems. There just two such cases. In the case of $\varphi = 0$ (the tilt is along the $\Gamma\Gamma_1$-line, i.e., in the direction of $b_1$) the system decouples into a two-level system and a four-level system, whereas in the case $\varphi = \pi/6$ (the tilt is in the direction of $b_1 + b_2$) the general system decouples into two three-level systems. Since the special cases could be relevant for future experiments, we consider them in detail.

**Special case $\varphi = 0$.** In this case we have $\lambda_1 = b|\alpha|$, $\lambda_2 = \lambda_1/2$ and $\lambda_3 = -\lambda_1/2$. Introducing new amplitudes $b_1$ by an orthogonal transformation, $b_1 = e_1$, $b_2 = (e_2 + c_6)/\sqrt{2}$, $b_3 = (e_3 + c_5)/\sqrt{2}$,
Fig. 7. Symmetric tunnelling through the \( \Gamma \)-point with the tilt directed along \( \mathbf{b}_1 \). As before and in Fig. 6 we use \( \varepsilon_1 = 3/2, \varepsilon_2 = 1/4, \varepsilon_3 = 1/2, I_0 = 0.1, \) and \( |\alpha| = 0.05 \).

\[ b_4 = c_4, b_5 = (c_2 - c_6)/\sqrt{2} \] \text{and} \[ b_6 = (c_3 - c_5)/\sqrt{2} \], we get for \( \mathbf{B}_0 = (b_1, b_2, b_3, b_4)^T \) a four-level system

\[
\begin{align*}
\dot{\mathbf{B}}_0 &= (\mathbf{A}_0 t + I_0 \mathbf{H}_0) \mathbf{B}_0, \\
\mathbf{H}_0 &\equiv \begin{pmatrix}
0 & \frac{\varepsilon_1}{\sqrt{2}} & \frac{\varepsilon_2}{\sqrt{2}} & \frac{\varepsilon_3}{\sqrt{2}} \\
\frac{\varepsilon_1}{\sqrt{2}} & \frac{\varepsilon_1 + \varepsilon_2}{2} & \frac{\varepsilon_2}{\sqrt{2}} & \frac{\varepsilon_3}{\sqrt{2}} \\
\frac{\varepsilon_3}{\sqrt{2}} & \frac{\varepsilon_2}{\sqrt{2}} & \frac{\varepsilon_1}{\sqrt{2}} & \frac{\varepsilon_2}{2} \\
\frac{\varepsilon_3}{\sqrt{2}} & \frac{\varepsilon_3}{\sqrt{2}} & \frac{\varepsilon_2}{\sqrt{2}} & 0
\end{pmatrix}
\end{align*}
\] (32)

with \( \mathbf{A}_0 = \text{diag}( -\lambda_1, -\lambda_1/2, \lambda_1/2, \lambda_1). \) The rest two amplitudes, \( b_5 \) and \( b_6 \), are decoupled from the system (32), they evolve according to the LZM system:

\[
\begin{align*}
\dot{b}_5 &= \left( -\frac{\lambda_1 t}{2} - \frac{\varepsilon_3 I_0}{2} \right) b_5 + \frac{I_0}{2} (\varepsilon_1 - \varepsilon_2) b_6, \\
\dot{b}_6 &= \left( \frac{\lambda_1 t}{2} - \frac{\varepsilon_3 I_0}{2} \right) b_6 + \frac{I_0}{2} (\varepsilon_1 - \varepsilon_2) b_5.
\end{align*}
\] (33) (34)

We see that Zener transitions take place between either four or two Bloch bands, depending on the initial populations, i.e., the initial values of \( b_j \). In the case of tunnelling between the two bands the probability is given by the LZM result \( P = \exp \left\{ -\frac{\pi I_0^2}{2\lambda_1} (\varepsilon_1 - \varepsilon_2)^2 \right\} \). More interesting is the case of tunnelling involving four bands with initial condition \( c_1(0) = b_1(0) = 1 \), so that, solving Eqs. (33)-(34), we obtain \( b_{5,6} = 0 \) and \( c_2 = c_6, c_3 = c_5 \). The probability of the transition \( P_{1 \rightarrow 1} = P_{4 \rightarrow 4} \) is known, and, in fact, follows from Eq. (30) which becomes \( P = \exp \left\{ -\frac{2\pi I_0^2}{\lambda_1} [\varepsilon_1^2 + \varepsilon_2^2/8 + \varepsilon_3^2/3] \right\} \). This case is illustrated in Fig. 6(b) and the results of the direct numerical simulations are presented in Fig. 7 and movie 6D.avi. It is clearly seen that tunnelling is indeed symmetric with respect to reflection in \( x \)-axis (i.e., \( c_2 = c_6, c_3 = c_5 \); \( x \)-axis corresponds to the \( \mathbf{b}_1 \)-direction).
Special case \( \phi = \pi/6 \). We have \( \lambda_1 = \frac{\sqrt{3}}{2} b |\alpha| \), \( \lambda_2 = \lambda_1 \) and \( \lambda_3 = 0 \). Therefore, in this case system (28) decouples into two three-level systems for the amplitudes given by the following orthogonal transformation \( b_1 = (c_1 + c_2)/\sqrt{2} \), \( b_2 = (c_3 + c_6)/\sqrt{2} \), \( b_3 = (c_4 + c_5)/\sqrt{2} \), \( b_4 = (c_1 - c_2)/\sqrt{2} \), \( b_5 = (c_3 - c_6)/\sqrt{2} \), \( b_6 = (c_4 - c_5)/\sqrt{2} \). We get:

\[
iB_j = \left( \Lambda_j z + \frac{I_0}{2} H_j \right) B_j, \quad j = 1, 2,
\]

with \( B_1 = (b_1, b_2, b_3)^T \), \( B_2 = (b_4, b_5, b_6)^T \), \( \Lambda_1 = \Lambda_2 = \text{diag}(-\lambda_1, 0, \lambda_1) \), and

\[
H_1 = \begin{pmatrix}
    e_1 & e_1 + e_3 & e_2 + e_3 \\
    e_1 + e_3 & e_2 & e_1 + e_3 \\
    e_2 + e_3 & e_1 + e_3 & e_1
\end{pmatrix}, \quad
H_2 = \begin{pmatrix}
    -e_1 & e_3 - e_1 & e_2 - e_3 \\
    e_3 - e_1 & -e_2 & e_1 - e_3 \\
    e_2 - e_3 & e_1 - e_3 & -e_1
\end{pmatrix}.
\]

In both cases, the probability of the transition \( P^{(j)} \equiv P^{(j)}_{1 \rightarrow 1} \equiv P^{(j)}_{3 \rightarrow 3} \) is known, it reads

\[
P^{(j)} = \exp \left\{ -\frac{\pi I_0^2}{2A_1} \left[ (e_1 + e_3)^2 + \frac{1}{2}(e_2 + e_3)^2 \right] \right\},
\]

with the plus sign for \( j = 1 \) and the minus for \( j = 2 \). For the initial condition \( c_1(-\infty) = 1 \) the tunnelling involves all six Bloch bands since in this case we have \( b_1(-\infty) = b_4(-\infty) = 1 \), see Fig. 6(c,d).

5. Vortex tunnelling

The theory developed in section 4 can also be extended to the interband transitions of optical beams carrying phase dislocations, e.g. optical vortices [46]. Consider, for instance, a Gaussian beam carrying a single vortex of the charge \( n \),

\[
\Psi_n = \frac{[(x-x_0) + iy-y_0]^n}{\sqrt{2\pi}\sigma} \exp \left\{ i k_0 (x-x_0) - \frac{(x-x_0)^2}{2\sigma^2} \right\}. \tag{37}
\]

The main point is based on the fact that a vortex in the real space corresponds to a vortex of the same vorticity (or topological charge) in the Fourier space according to the rule

\[
\mathcal{F}\{\Psi_n\}(k) = i [\partial_k - \partial_n - (x_n + iy_n)]^n \mathcal{F}\{\Psi_0\}(k), \tag{38}
\]

e.g. the Gaussian beam Eq. (37) in the Fourier space reads

\[
\mathcal{F}\{\Psi_n\} = \sqrt{2\pi}\sigma (-i\sigma^2)^n [(k-k_0) + i(\lambda - \lambda_0)]^n \exp \left\{ -i x_0 (k-k_0) - \frac{\sigma^2}{2}(k-k_0)^2 \right\}, \tag{39}
\]

where \( k = k n_x + \lambda n_y \). In the absence of the lattice, but with the linear potential alone, \( V_{\text{lin}} = ax \), the wave (37) expands (diffracts), \( \sigma^2(t) = \sigma^2 + it \), acquiring a chirp and an overall phase according to the law \( x_0(t) = x_0 + \int_0^t k_0(t)dt \), \( k_0(t) = k_0 - at \). Note that a vortex initially placed close to the beam center \( x = x_0 \) remains close to \( x_0(t) \) for all times.

First, we demonstrate that the Zener tunnelling of a beam carrying a phase dislocation in a periodic lattice results in several output beams with the phase dislocations of the same charge. As an example, we consider the case of a single \( n \)-vortex in a Gaussian beam, i.e., the \( n \)-order Laguerre-Gauss beam. The Bragg resonance will result in several identical Gaussian peaks in the Fourier space with the indices shifted by the resonant reciprocal lattice vectors. For instance, near the resonance at \( t = 0 \), for the output Fourier amplitude with the index \( q - Q \) we have

\[
C(q - Q, t) = -i\tilde{V}_q C(q, 0) t + O(t^2),
\]
Fig. 8. Three-fold symmetric Zener tunnelling of a vortex beam, see the movie V-Lavi. (a) Intensity and (b) phase of the initial beam with the topological charge $m = -1$ in the vicinity of the M-point, see (c). Intensities in the Fourier domain (c,d) are identical for both, vortex, $m = +1$, and antivortex, $m = -1$, while they differ significantly in the real space, compare (e,g) and corresponding phases (f,h).

where $C(q,0)$ is the Fourier image of the input wave (in this case, $q = k_0$, it has the Fourier index inside the first Brillouin zone). Therefore, from Eq. (11) and Eq. (39) we conclude that every output peak in the Fourier space tagged by $q - Q$ will have an $n$-vortex placed on it at its center index $k = k_0(t) - Q$. It is important to note that the position of the output beam $C(q - Q)$, $x_0(t) = x_0 + (k_0 - Q)t - \alpha \sigma^2/2$, is carrier-dependent through $Q$. Returning to the real space, we obtain an $n$-vortex sitting on each of the output waves (whose velocities are different). Thus the phase dislocations resulting from the tunnelling are identical and preserved in the evolution between the resonances. The vortices can be replaced by anti-vortices without any affect on the distribution of power in the Fourier space.

We have performed numerical simulations using the Laguerre-Gaussian vortex beams for the case of symmetric three-fold resonance, similar to Fig. 4 and [24]; the results are presented in Fig. 8. In the Fourier space, the intensity distribution does not depend on the vortex topological charge, see Fig. 8(c,d), and the vortices have the same charge as the initial beam. We do not observe any difference in the tunnelling efficiency, simply repeating the results with the Gaussian beams in Fig. 4. In the real space, however, due to interference, the two solutions corresponding to the tunnelling of vortices and that of anti-vortices are substantially different, see Fig. 8(e,g), since the phases of the carriers are the same while the vortex phases are conjugated, when a vortex is replaced by an anti-vortex. At the same time, the phase of the tunneled vortex beam is simply of the opposite charge, see the characteristic fork-type phase dislocations in Fig. 8(f,h) similar to the one in Fig. 8(b). Note also that the dynamics of anti-vortices is equivalent to that of vortices in the complex conjugate Schrödinger equation, which requires the inversion of time $t \to -t$.

6. Conclusions

We have analyzed the interband transitions, such as Zener tunnelling and Rabi oscillations, in hexagonal photonic lattices and derived several different multi-level LZM systems that capture the essence of these phenomena. We have demonstrated that the direction of the tilt together
with the Fourier coefficients of the lattice potential determine how many of the lowest-order Bloch bands of the photonic bandgap spectrum are involved in the interband transitions. We have identified three general regimes of the Zener tunnelling in the hexagonal photonic lattice: (i) quasi one-dimensional Zener tunnelling (or, equivalently, simple Bragg resonance involving only two Bloch bands) which occurs when the Bloch index crosses the Bragg planes far from one of the high-symmetry points; (ii) three-fold Bragg resonance at the high-symmetry M-point with the Zener transitions between the three Bloch bands; and (iii) the six-fold Bragg resonance at the high-symmetry Γ-point with the Zener tunnelling involving, in general, six Bloch bands. For some special directions of the tilt, the tunnelling through the Γ-point is described by either two three-level systems or by a combination of one four-level and one two-level systems; the six Bloch bands decouple into two groups of either two pairs of three bands or four bands and two bands, respectively, with the Zener tunnelling taking place only between the bands of the same group. In the special symmetric cases, we have found reductions of the general LZM models and found several interesting features of the tunneling dynamics, for instance, we have found a new effect, which can be called Zener tunneling of Rabi oscillations. In addition, we have shown that tunnelling of phase dislocations (or optical vortices) results in the output waves carrying the same phase dislocations. In the real space, there is an asymmetry in the interference pictures resulting from the tunnelling of vortices and anti-vortices, since the dynamics of vortices is equivalent to that of anti-vortices in the complex-conjugated equation with the inverted time.

Our theoretical results have immediate applications to a number of different phenomena such as the light beam propagation in optically-induced photonic lattices, tunnelling of Bose-Einstein condensates in optical lattices, and other types of wave propagation in tilted hexagonal periodic potentials. Our results may also provide an additional field for the search for the integrable LZM systems (see, e.g., [35, 36, 37] and references therein). Indeed, one would expect that the multi-mode LZM systems obtained from the highly-symmetric hexagonal lattices are integrable, in the sense that they allow an analytical expression for the S-matrix (as defined in [35]), so that the transition probabilities between all levels could be found in an analytic form.

We should notice, however, that in the derivation of the LZM models, we have neglected the effects of nonlinearity since our aim was to elucidate the role played by the lattice symmetry on Zener interband transitions. In the case of optical beams propagating in the photonic crystals, the effect of nonlinearity can be negligible indeed, while nonlinearity of BEC has been found to cause the breakdown of adiabaticity in the Zener tunnelling [28] and change the symmetry between the upper-to-lower vs. lower-to-upper interband transitions [30, 28, 29, 31]. Nonlinear tunnelling in the hexagonal photonic lattices will be considered in one of our future publications (see also Ref. [32] where the nonlinear tunnelling has been considered for the case of a square lattice).

Acknowledgments
The work has been supported by the CAPES and FAPESP of Brazil (VSS) and the Australian Research Council in Australia (ASD and YSK). VSS also thanks the Nonlinear Physics Center of the Australian National University for support and hospitality.