Ideal and nonideal invisibility cloaks

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Abstract: We employ the analytical solutions for the spatial transformation of the electromagnetic fields to obtain and analyze explicit expressions for the structure of the electromagnetic fields in invisibility cloaks. Similar approach can be also used for analyzing beam splitters and field concentrators. We study the efficiency of nonideal electromagnetic cloaks and discuss the effect of scattering losses on the cloak invisibility.

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Cloaking of objects was suggested by Pendry et al. [1, 2] as a method to cover an object by a composite material with varying characteristics in such a way that the scattering of electromagnetic waves from this object vanishes, so that the electromagnetic field outside the cloak appears as if the object and cloak are being absent. The required material for creating such invisibility cloaks should have rather complex spatially varying properties. Moreover, such materials should possess both nontrivial dielectric and magnetic responses. Generally speaking, it is impossible to find in nature a dielectric medium with the required properties, so that the first electromagnetic cloak realized experimentally employed microwave metamaterials [3]. Metamaterials are artificial micro-structured materials with electric and magnetic properties designed by careful engineering of their constituents. The constituents of these composite materials are normally sub-wavelength electric and magnetic resonators, with the specific properties determined by their geometry. The metamaterials can also be anisotropic, and thus they can satisfy, in principle, all the requirements for realizing invisibility cloaks for electromagnetic waves.

The electromagnetic cloaking is based on the coordinate transformation of space and the corresponding transformation of Maxwell's equations, which provide the required expressions for the effective spatially varying dielectric permittivity and magnetic permeability [1]. The possibility of creating such cloaks has been confirmed experimentally [3], and it was further discussed and demonstrated in many numerical studies [4, 5, 6, 7, 8, 9, 10, 11, 12]. Numerical simulations are usually performed either by using commercial finite-element equation solvers [1, 8, 9, 10, 11, 12] or by employing the decomposition of the electromagnetic fields into a set of the corresponding eigenmodes [4, 5, 6, 7, 13]. Several studies have analyzed a possibility of creating simplified cloaks [8, 10, 12], since the required media parameters for an ideal cloaking are extremely complex for their realization in experiment. In particular, an optimization of the cloaking parameters for suppressing scattering has been analyzed in Refs. [6, 10]. In this context, we mention earlier studies [14, 15], where an analogy of the coordinate transformation to an effective magneto-dielectric medium was discussed for some particular cases.

In this paper we use the explicit analytical expressions for calculating the structure of the electromagnetic fields modified by spatially varying coordinate transformations. Based on our results we evaluate the efficiency of nonideal cloaks and calculate their scattering properties.

We start by considering the propagation of monochromatic (i.e. $\sim \exp(i\omega t)$) electromagnetic wave in empty space. In the following we use the component notations. Let $\mathbf{x}_a$ is the orthonormal basis corresponding to the Cartesian coordinates, the indices (a,b,c) run through the values $(1,2,3)$, and x^a are the Cartesian coordinates corresponding to the orthonormal basis $\mathbf{x}_a$. In a medium with the dielectric permittivity $\hat{\varepsilon}_0$, the electric displacement vector $\mathbf{D}$ can be expressed in terms of the electric field as $D^a = \hat{\varepsilon}_0^{ab} E_b$. The magnetic displacement vector $\mathbf{B}$ can be expressed as $B^a = \hat{\mu}_0^{ab} H_b$, in a medium with magnetic permeability $\hat{\mu}_0$. We note that the repeated upper and lower indices imply summation (Einstein convention).

Now we introduce new coordinates $\mathbf{y}$ through the transformations $\mathbf{y} = \mathbf{y}(\mathbf{x})$ with the metric tensor g^{ij} defined in a standard way,

$$g^{ij}(x) = \frac{\partial y^i}{\partial x^a}\frac{\partial y^j}{\partial x^b}\delta^{ab} , \qquad (1)$$

where δ^{ab} is Kronecker's delta-function. The determinant $g(x) = \det||g^{ij}(x)||$ is of a particular importance for the coordinate transformations. It is known that Maxwell's equations are invariant (preserve their formal structure) for any continuous coordinate transformation $\mathbf{y} = \mathbf{y}(\mathbf{x})$ if

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dielectric and magnetic tensors are modified according to

$$\varepsilon_{\rm eff}^{ln} = \frac{1}{\sqrt{g}} \frac{\partial y^l}{\partial x^a} \frac{\partial y^n}{\partial x^b} \varepsilon_0^{ab} \,, \tag{2}$$

$$\mu_{\rm eff}^{ln} = \frac{1}{\sqrt{g}} \frac{\partial y^l}{\partial x^a} \frac{\partial y^n}{\partial x^b} \mu_0^{ab} . \tag{3}$$

These expressions are widely used in the literature for electromagnetic cloaks and other devices. Corresponding transformation formulas for the electric and magnetic fields are

$$\tilde{E}_n = \frac{\partial x^b}{\partial y^n} E_b \,, \; E_b = \frac{\partial y^n}{\partial x^b} \tilde{E}_n \,. \tag{4}$$

$$\tilde{H}_k = \frac{\partial x^c}{\partial y^k} H_c, \quad H_c = \frac{\partial y^k}{\partial x^c} \tilde{H}_k. \tag{5}$$

These equations allow us to find analytically the field distribution for any cloak *without* solving Maxwell's equations in the transformed geometry, but only transforming the free-space fields. As a straightforward application, these expressions allow for analytical calculation of the scattering from non-ideal cloaks, and thus a potential optimization of their parameters.

We can use those results for calculating the field structure in both cylindrical and spherical cloaks, field concentrators, and other devices designed via the transformation optics. As the first example, we consider the linear coordinate transformation for the cylindrical cloak [18], $X = X(x,y)$, $Y = Y(x,y)$, $Z = z$, where the radius is transformed as $R = a + r(b-a)/b$, where $R = \sqrt{X^2+Y^2}$, $r = \sqrt{x^2+y^2}$. The cloak occupies the space $a < R < b$, where a and b are the inner and outer radii of the cloak, respectively. To find the field distribution in the cloak illuminated by a plane electromagnetic wave, we write the plane wave in vacuum as $H_x = \exp(i\kappa_y y)$, $H_y = 0$, $E_z = \exp(i\kappa_y y)$. Applying now the transformations according to Eqs. (5), we find the field distribution in the electromagnetic cloak with the material parameters (2) and (3), $\tilde{E}_z = \exp(i\kappa_y Y)[b\,(b-a)^{-1}\,(1-aR^{-1})]$, $\tilde{H}_x = b(b-a)^{-1}(1-aY^2R^{-3})\tilde{E}_z$, $\tilde{H}_y = abXY(b-a)^{-1}R^{-3}\tilde{E}_z$, for $a < R < b$. The field vanishes inside the cloaked area, whereas outside of the cloak the field is an unperturbed plane wave. At the external surface of the cloak (at $R = b$), the normal component of the magnetic field is discontinuous, since the linear space transformation produces discontinuities of the cloak material parameters. Using the transformation function

$$r = b - a\left(\frac{b-R}{b-a}\right)^{\beta} \tag{6}$$

provides continuity of the field components at the external surface of the cloak for large β.

As a matter of fact, Eqs. (5) and (4) allow us to find the field structure for *an arbitrary continuous space transformation*. Though for an arbitrary transformation one cannot find the analytical expressions for the electromagnetic fields, simple numerical calculations provide the required results without solving Maxwell's equations.

As an example of a cloak of complex shape, we consider cut-type space transformation where the cloak covers an arbitrary area 'cut' from the space (see, e.g., Fig. 1). We define the internal surface of the cloak as a set of all points which are closer than b to the cut line $X_{\rm cut}(\xi), Y_{\rm cut}(\xi)$. For each point (X,Y) we find the corresponding closest point on the cut line $(X^0_{\rm cut}, Y^0_{\rm cut})(X,Y)$ by finding the minima of the function $[(X - X_{\rm cut}(\xi))^2 + (Y - Y_{\rm cut}(\xi))^2]^{1/2}$ for all possible ξ. We define the distance from the 'cut' as $R(X,Y) = [(X - X^0_{\rm cut})^2 + (Y - Y^0_{\rm cut})^2]^{1/2}$ and create the cloaking area $a < R < b$ by transforming the space according to Eq. (6). In the area of

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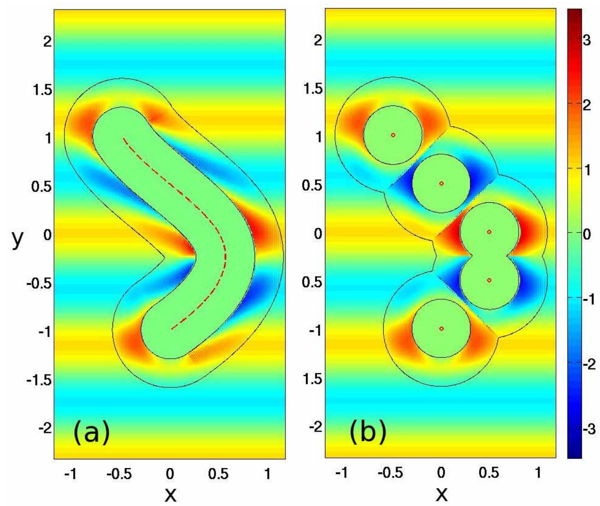

Fig. 1. Structure of the magnetic field H_x for (a) arbitrary-shaped cloak and (b) multiple cloaks produced by the space transformation Eq. (6) for $a = 0.3$, $b = 0.6$, and $\beta = 2$. The cloak is illuminated by a plane wave, $H_x^0 = \exp(i\kappa_y y)$ for $\kappa_y = 5.4$. Center of the cloak is shown by red line, in (a), and dots, in (b).

the cloak, $a < R < b$, we define the new coordinates (x, y) as $x = X_{cut}^0 + (X - X_{\mathrm{cut}}^0)(r/R)$ and $y = Y_{cut}^0 + (Y - Y_{\mathrm{cut}}^0)(r/R)$, which provide mapping of (x, y) into (X, Y).

Figure 1(a) shows the distribution of the x-component of the magnetic field for the plane wave incident on the cloak. The cut is created through the graphic input in Matlab, where we specify several points (five points, in this example), which are connected by a smooth line through the cubic spline function. Figure 1(b) shows the field distribution in the cloaking problem with several cylindrical cloaks centered at the same five points.

We note that, despite the divergence of the dielectric and magnetic functions at the internal interface of the cloak, the electric and magnetic fields remain finite. For example, the radial and tangential components of the magnetic field can be found as

$$|H_R| = \frac{b}{(b-a)}\frac{|X|}{R}\ ,\ |H_\tau| = \frac{b}{(b-a)}\frac{|Y|}{R}\left(1 - \frac{a}{R}\right).$$

We note that $|H_R|$ is finite at the internal interface of the cloak, while $|H_\tau|$ vanishes when $R \to a$.

From the uniqueness theorem of electromagnetism we can conclude that since the tangential components of the electric or magnetic field at the closed surface vanish, the field in the volume surrounded by this surface vanishes as well; this means that we indeed have the volume ($R < a$) concealed from the electromagnetic radiation.

Above we demonstrated that a cloak with the parameters defined by Eqs. (2) and (3) provides complete invisibility of the objects hidden inside. Moreover, such a cloak works for an arbitrary transformation function $\mathbf{y} = \mathbf{y}(\mathbf{x})$, either smooth or piece-wise continuous [8]. However, this kind of ideal cloaking seems to be impossible in practice, since it requires infinite values of material parameters at the internal surface. The simplest way to overcome this difficulty is to truncate the transformed area. For example, for the coordinate transformation with the internal radius a we create the cloak only for $R > a + \delta a$, and then place a perfectly conducting metal surface at $R = a + \delta a$. Now, if we make an inverse transformation from the cloak to vacuum,

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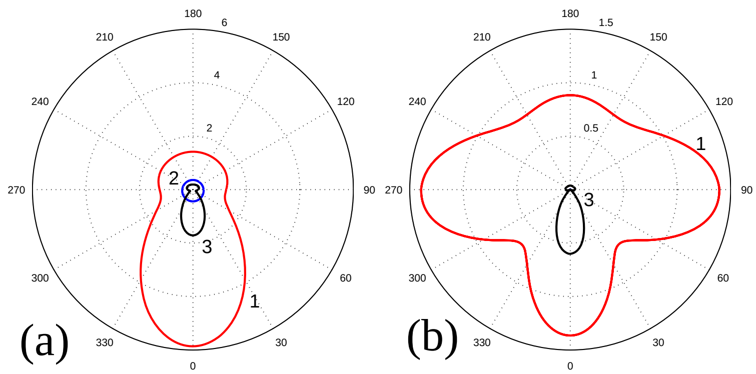

Fig. 2. (a,b) Scattering patterns for non-ideal cloak for TE and TM waves. Red lines (1) indicate scattering from bare metallic cylinder, while blue line (2) corresponds to the scattering from the cloaked cylinder with truncation level $r_0 = 0.05b$ and $\alpha = 0$. Black curves (3) correspond to the scattering on the dielectric cloak with (a) $\alpha = 0.1$ and (b) $\alpha = 0.02$.

the metallic surface transforms into a metallic cylinder with the radius r_{eff} where

$$r_{\text{eff}} \approx \delta a (dR/dr|_{r=a})^{-1}, \tag{7}$$

provided $dR/dr|_{r=a} \neq 0$, i.e. *the simplified cloak scatters light as a metallic cylinder* of the radius r_{eff}, which can be much smaller than the size of the hidden region a.

One of the main limitations of the resonant medium used for creating electromagnetic cloaks is losses. To estimate the effect of small losses on the cloak performance, we assume that $\hat{\varepsilon} = \hat{\mu} = \hat{\varepsilon}' + i\hat{\varepsilon}''$, $\hat{\varepsilon}'' \ll \hat{\varepsilon}'$, whereas $\hat{\varepsilon}'$ is given by Eq. (2). We consider a plane wave incident on such a cloak. After applying the inverse coordinate transformation, which transforms the electromagnetic fields into vacuum plane waves, we obtain in original space the excited effective electric and magnetic currents dependent on vacuum electric and magnetic fields,

$$j^a_{e(m)} = \sigma^{ab} E^v_b (H^v_b), \;\; \sigma^{ab} = -\frac{\omega\sqrt{g}}{4\pi}\frac{\partial x^a}{\partial y^i}\frac{\partial x^b}{\partial y^j}\varepsilon^{ij\prime\prime}.$$

These currents lead to the scattering radiation which can be easily evaluated either analytically or numerically.

Now, we can estimate the effect of mismatched dielectric permittivity and magnetic permeability on the cloak performance. We assume that $\hat{\varepsilon} = \hat{\mu} + \delta\hat{\varepsilon} = \hat{\varepsilon}_0 + \delta\hat{\varepsilon}$, where $\hat{\varepsilon}_0$ corresponds to an ideal cloak and $|\delta\hat{\varepsilon}| \ll |\hat{\varepsilon}_0|$. Then, we apply the inverse transformation $\mathbf{x} \to \mathbf{y}$ and find that in the original space the dielectric permittivity differs from the unity by the value $\delta\tilde{\varepsilon}^{ab} = \sqrt{g}(\partial x^a/\partial y^i)(\partial x^b/\partial y^j)\delta\varepsilon^{ij}$. If we assume that $(\hat{\varepsilon} - \hat{\mu}) \propto \hat{\varepsilon}$, namely $\delta\varepsilon^{ij} = \alpha\varepsilon_0^{ij}$ (for $\alpha \ll 1$), then $\delta\tilde{\varepsilon}^{ab} = \alpha\delta^{ab}$. Thus, in this case the non-ideal cloak scatters the waves as an object (cylinder of the radius r_{eff} for the cylindrical cloak) with a scalar dielectric permittivity $1 + \alpha \approx 1$.

Now we consider $\delta\varepsilon^{ij} = \alpha\delta^{ij}$, for $\alpha \ll 1$. This gives us a relatively simple result for $\delta\tilde{\varepsilon}^{ab}$,

$$\delta\tilde{\varepsilon}^{ab} = \alpha\sqrt{g}\frac{\partial x^a}{\partial y^i}\frac{\partial x^b}{\partial y^i} \tag{8}$$

and also an expression for the electric current

$$\mathbf{j}_{\text{eff}} \approx \frac{i\omega}{4\pi}\alpha\hat{\varepsilon}^{-1}\mathbf{E}^v, \tag{9}$$

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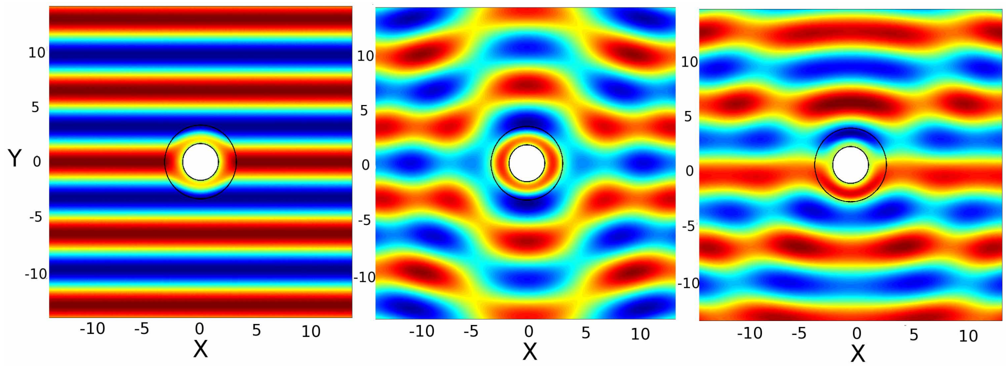

Fig. 3. Field distribution for TE waves for (a) ideal cloak; (b) for impedance mismatched cloak with $\alpha = 0.03$, (c) for lossy cloak with $\alpha = i \times 0.03$

which can be used for calculating the scattered fields. It is worth mentioning that in this case the cloak is sensitive to small perturbations of the parameters. Indeed, the inverse dielectric permittivity has a singularity at the inner surface of the cloak, so if the truncation value δa is small then the parameter α has to be very small as well so that it does not perturb the structure of the plane wave. This result has been predicted earlier in Ref. [5]. We also note that very similar expressions for the electric and magnetic currents in the lossy case may also lead (for particular forms of $\hat{\varepsilon}''$) to such a strong sensitivity.

As an example, we consider the plane wave scattering on a slightly non-ideal cloak with the magnetic permeability tensor components $\mu_{rr} = (R-a)/R$, $\mu_{\phi\phi} = R/(R-a)$, $\mu_{zz} = \gamma^{-2}(R-a)/R$, $\gamma = (b-a)/b$, and dielectric permittivity tensor $\varepsilon_{ii} = \mu_{ii} + \alpha$, $\alpha \ll 1$. We assume that the wave propagates in the y-direction with $k_y = 5.4$, and the inner and outer radii of the cloak are $a = 0.3$ and $b = 0.6$, respectively. For $\alpha = 0$ this is an ideal cloak, and it corresponds to the linear coordinate transformation $R = a + \gamma r$, $0 < r < b$. In order to avoid the singularity we place a metallic cylinder at $R = a + \delta a$, which conceals the inner cloak region but also distorts scattering performance. Scattered fields are regarded as weak, so that the perturbation approach can be used. We make an inverse coordinate transform $R \to r$, so that the metal screen radius transforms to $r_0 = \delta a/\gamma$, then calculate the effective current according to Eq. (9) and find the scattering field pattern for both TE ($\mathbf{E}^0 = \mathbf{z}^0 E$) and TM ($\mathbf{H}^0 = \mathbf{z}^0 H$) waves [see Figs. 2(a,b)].

As a result, for the TM waves the scattering from non-ideal cloak is negligible, and in the shown scale it is represented by a point in the center of Fig. 2(b). If the truncation level r_0 is reduced, the scattered radiation for TM waves is also reduced. On the contrary, the TE wave scattering is significantly enhanced, e.g., it becomes 4 times larger for $r_0 = 0.005b$.

This approach allows estimating scattering losses in arbitrary nonideal cloaks (e.g. such shown in Fig. 1), since the effective currents can be calculated using Eq. (9). Figure 3 shows distributions of the field for ideal cloak, impedance-mismatched cloak and for a lossy cloak calculated for TE waves using our approach. For TM waves, when electric field has just one component parallel to the cylinder axis, scattering is much stronger, which again demonstrates strong difference in the effect of the nonideal cloak on different polarizations.

In conclusion, we have employed the analytical results for the structure of electromagnetic fields in the transformed space to analyze the problem of electromagnetic cloaking and derive a simple criterion of the efficiency of nonideal cloaks. The analytical formulas can be useful for other devices of the transformation optics such as beam splitters and energy concentrators.

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