THREE-PARTICLE AND INELASTIC EFFECTS IN THE INTERACTION OF CONSERVATIVELY PERTURBED SINE-GORDON EQUATION KINKS

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Received 25 September 1985; accepted in revised form 6 March 1986

Three-particle effects in the collision of a fast kink with two more slow ones described by the conservatively perturbed sine-Gordon equation are considered. The collision of a fast kink with a more slow one in the presence of a localized inhomogeneity, and radiative effects accompanying the reflection of a kink from the edge in a problem on a half-line are considered too.

1. This paper, which is closely related to the preceding one [1], is devoted to the investigation of three-particle and inelastic effects within the framework of the sine-Gordon equation (SGE) perturbed by a small conservative term. First we shall consider the equation [2]

\[ \frac{\partial^2 u}{\partial t^2} - u_{xx} + \sin u = \epsilon \sin(2u), \quad \epsilon \ll 1. \quad (1) \]

The simplest soliton solution to the unperturbed SGE is the kink:

\[ u(x, t) = 4\sigma \arctan\{ \exp[-\nu(x - Vt - x(0))] \}, \quad \sigma = \pm 1 \quad (2) \]

where \( \sigma = \pm 1 \) is the soliton's polarity, \( V \) is its velocity, and \( \nu = (1 - V^2)^{-1/2} \). As in the case of the conservatively perturbed nonlinear Schrödinger equation dealt with in ref. [1], a collision of two kinks in the perturbed SGE (1) does not result, in the first order of perturbation theory, in energy and momentum exchange between the kinks. We shall demonstrate that the exchange is possible as a three-particle effect generated by a collision between the fast kink with velocity \( V_3 \), such that \( V_3 \gg 1 \), and a pair of two slower ones separated by a distance \( D = x_1^{(0)} - x_2^{(0)} \) at the collision moment \( t = 0 \). We shall consider the collision in the reference frame where the velocities of the slow kinks are \( V_{1,2} = \pm W \). The essential presumptions, analogous to eqs. (3), (4) of ref. [1], are

\[ 1 - V_3^2 \ll 1 - W^2 \ll 1. \quad (3) \]

The relation between the small quantities \( 1 - V_3^2 \), \( 1 - W^2 \) and the underlying parameter \( \epsilon \) is the same as in ref. [1]: \( \epsilon \) is the smallest quantity, so that the contribution from the second order of the perturbation theory to all the results obtained below under the condition (3) is negligible.

The perturbation-induced evolution of the kink's velocity is determined by the well-known equation (see, e.g., ref. [2])

\[ \frac{d(\lambda_n)}{dt} = \frac{i}{4} \epsilon C_n(t) \int_{-\infty}^{\infty} dx \sin[2u(x, t)] Q_n(x, t), \quad (4) \]

(cf. eq. (5) of ref. [1]), where

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\[ Q_n(x, t) = [\Psi_n^{(1)}(x, t)]^2 - [\Psi_n^{(2)}(x, t)]^2, \quad (5) \]
\[ \lambda_n = \frac{1}{2}(1 + V_n)^{1/2}(1 - V_n)^{-1/2}. \quad (6) \]

Here \( \Psi_n^{(1,2)}(x, t) \) stand for the two components of the kink Jost function, and
\[ C_n(t) = -2a_n \lambda_n \]
\[ \times \exp\left[ (\lambda_n + 1/4\lambda_n)x^{(0)}_n + (\lambda_n - 1/4\lambda_n)t \right]. \quad (7) \]

As well as in refs. [1,2], in the first approximation the conditions (3) provide the splitting of the full wave potentials: \( u = u_1 + u_2 + u_3 \). As for the Jost functions, it is demonstrated in ref. [2] that they do not split, but their quadratic combination (5), essential for (4), does split, i.e. under the conditions (3) one may insert \( Q_n(x, t) \) into (4) in the “one-particle” approximation for each \( n \).

The total changes \( \Delta \lambda_n \) of the kinks’ parameters can be found as
\[ \Delta \lambda_n = \int_{-\infty}^{\infty} dt \frac{d\lambda_n}{dt} \quad (8) \]
(cf. eq. (7) of ref. [1]). As in ref. [1], the condition (3) together with the above splittings result in separating the time and spatial integrations in (8) and (4). Both integrations become elementary to yield
\[ \Delta \lambda_1 = \frac{128}{9} \nu_3^{-1} e g(\delta), \quad (9) \]
where the odd function
\[ g(\delta) = \tanh \delta \sech^2 \delta (1 - 2 \sech^2 \delta), \quad (10) \]
(cf. the function (14) from ref. [1]), and
\[ \delta = D/2\lambda_1 \quad (11) \]
characterizes the degree of overlapping between the slower kinks at the moment of the collision with the fast one. For the second kink analogous considerations results in
\[ \Delta \lambda_2 = -\frac{8}{3} \nu_3^{-1} \lambda_1^{-4} g(\delta). \quad (12) \]
As, according to (3) and (6), \( \lambda_1 \gg 1 \), we see from comparing (12) and (9)
\[ |\Delta \lambda_2| \ll |\Delta \lambda_1|. \quad (13) \]
This discrimination is related to the fact that the first kink has a velocity coinciding in direction with the fast kink’s velocity while the second kink’s velocity is opposite to that of the fast kink, i.e., with regard to (3), the effective time of interaction with the fast kink is much larger for the first slow kink than for the second one.

The change of the fast kink’s velocity can be easily found with the aid of energy and momentum conservation. Indeed, the energy \( \Delta E \) and momentum \( \Delta P \) transferred by the fast kink to the slower ones are
\[ \Delta E = 8 \sum_{n=1,2} (1 - 1/4\lambda_n^2) \Delta \lambda_n \approx 8 \Delta \lambda_1, \quad (14) \]
\[ \Delta P = 8 \sum_{n=1,2} (1 + 1/4\lambda_n^2) \Delta \lambda_n \approx 8 \Delta \lambda_1 \approx \Delta E, \quad (15) \]
we have used (9), (12) and (13) for neglecting the second terms in (14) and (15). The change of the fast kink’s parameter \( \lambda_3 \) can be found from (14):
\[ \Delta \lambda_3 = -\frac{128}{9} \nu_3^{-1} g(\delta). \quad (16) \]
To verify the result (16) we have also derived it directly from (4), without using the conservation laws.

As well as in ref. [1], the function \( g(\delta) \) defined in (10) vanishes when \( \delta = \infty \) which purports the three-particle character of the considered effect. It is also interesting to note that this function is odd, i.e., depending on the sign of the parameter \( \delta \), the fast kink may both lose and acquire energy due to the collision with the pair of the overlapping slow kinks. At last, it is worth noting that, as is seen from (9)–(11), the effect does not depend on the polarities of the kinks involved.

2. Now we proceed to another perturbed SGE [2, 3]
\[ u_{tt} - u_{xx} + \sin u = e\delta(x) \sin u, \quad (17) \]
which describes, e.g., a microinhomogeneity in the Josephson junction [3]. The collision of two kinks described by (17), and application of this problem to the Josephson junction theory were discussed in ref. [5]. From our viewpoint, the collision can generate three-particle effects since the inhomogeneity plays the role of the third “particle”. The results take a simple form in two particular cases. First we shall consider the collision of a fast kink with a slower one in the presence of the microinhomogeneity (17). Following
the lines set forth above, we obtain the collision-induced changes $\Delta \lambda_3$, $\Delta \lambda_1$ of the parameters (6) of the fast and slow kinks:

$$\Delta \lambda_3 = -2eV_1 \nu_1 \nu_3^{-1} g_1(\delta) ,$$  

$$\Delta \lambda_1 = 2e\lambda_1 \nu_3^{-1} g_1(\delta) ,$$

$g_1(\delta)$ being the same function as in ref. [1]. The parameter $\delta = \nu_1 (x_1^{(0)} - V_1 x_3^{(0)})$ characterizes the overlapping between the slow soliton and the inhomogeneity at the moment of collision $t = 0$. It is easy to verify that, according to (18), (19), the total energy of the two kinks

$$E = 8 \sum_{n=1,3} (\lambda_n + 1/4\lambda_n)$$

is conserved, while the momentum $\Delta P$ absorbed by the inhomogeneity is

$$\Delta P = 8e\nu_1^{-1}(1 - V_3^2)^{1/2} g_1(\delta) .$$

Note that (20), as well as (10), is odd with respect to $\delta$.

The second explicitly tractable case is that when the inhomogeneity is in rest in the center-of-mass reference frame of the two colliding kinks, i.e. their velocities are opposite: $V_1 = -V_3 = \nu_3 (\text{now we do not demand } \nu^{-1} = (1 - W^2)^{1/2} < 1)$. The result is

$$\Delta \lambda_2 = -\Delta \lambda_1/4 \lambda_1^2 ,$$

$$\Delta \lambda_1 = -e_\omega W(1 + W)D_+ D_- [6 + (4\sigma - b)J_\sigma(W, \delta)] / (2(2\sigma + b)^2) ,$$

where

$$J_\sigma(W, \delta) = (2/\sqrt{-\Delta}) \ln[1/2(b + \sqrt{-\Delta})] , \quad \Delta < 0 ,$$

$$= (2/\sqrt{\Delta}) \arccot(b/\sqrt{\Delta}) , \quad \Delta > 0 ,$$

and

$$\Delta = -W^2 D_+^2(2\sigma + b) , \quad b = 2\sigma + W^2 D_-^2 ,$$

$$D_+ = e^\delta \pm ae^{-\delta} .$$

Here $\sigma = \sigma_1 \sigma_3$ is the relative polarity; $\delta = x_0/(1 - W^2)^{1/2}$, where $x_0$ is the distance of the inhomogeneity from the center of mass in rest of the two-kink system. Note that in this case, as is seen from (21), $\Delta P = 0$, contrary to (20), i.e. the inhomogeneity in rest in the center-of-mass system in the first approximation cannot absorb momentum (the momentum nonconservation is possible in higher approximations unless $x_0 = 0$).

The above results are equally applicable to the cases $\epsilon > 0$ and $\epsilon < 0$. At the same time it is well known [3] that a kink may be pinned (bound) by the inhomogeneity only in the case $\epsilon > 0$, when the inhomogeneity attracts kinks of both polarities, the attraction potential being

$$U = -2e \sech^2 \xi ,$$

where $\xi$ is the coordinate of the kink’s center. Using energy conservation and neglecting radiation emission, it is easy to find the law of motion for the bound kink [3]:

$$\sinh t(\xi) = \sqrt{(2e - E)/E} \sin(\sqrt{E} t/2) ,$$

$E$ being the binding energy ($0 < E < 2e$). The collision of a fast kink with a bound one described by (23) may result in inelastic effects analogous to those dealt with in ref. [2], but which is much simpler to treat: kicking out the slow kink from the bound state. If the incident kink is not too slow, we may use the expression for the collision-induced phase shift $\Delta \xi$ of the slow (bound) kink, valid in the absence of perturbation [4]:

$$\Delta \xi \approx -\log \left( \frac{1 + (1 - V_3^2)^{1/2}}{1 - (1 - V_3^2)^{1/2}} \right) ,$$

$V_3$ being, as above, the fast kink’s velocity. The collision duration is $\sim V_3^{-1}$, what is much smaller than the bound kink’s oscillation period $4\pi/\sqrt{E} \sim e^{-1/2}$ (see (23)), hence we may neglect the change of the bound kink’s velocity during the collision. Thus the collision does not change the slow kink’s kinetic energy, while, according to (22) and (24), its potential energy acquires a change $\Delta U = U(\xi_0 + \Delta \xi) - U(\xi_0)$, where $U(\xi)$ and $\Delta \xi$ are defined in (22) and (24), $\xi_0$ being the instantaneous value of the slow kink’s coordinate before the collision. So, it is evident that the bound kink escapes provided $\Delta U > E$, and it remains bound in the opposite case. This consideration and the final result are valid provided $V_3 > \sqrt{E}$.

In the opposite case, when $V_3$ is small, another noticeable interaction may occur: reflection of the incident kink from the bound one, without releasing
the latter kink, provided the kinks are unipolar. Note that the inhomogeneity itself is attractive, and it cannot reflect a kink. The maximum (threshold) value of $V_3$ for which the reflection is possible can again be readily obtained by means of the energetic arguments: 

$$V_{\text{thr}}^2 = \frac{3}{4}E.$$ 

3. Eq. (1) on the half-line $x \geq 0$ with the boundary conditions 

$$u_x(x = 0) = 0 \quad \text{or} \quad u(x = 0) = 0 \quad (25a, b)$$ 

has applications, e.g., in one-dimensional magnetodynamics ((25a) and (25b) correspond to half-infinite spin chains with free and pinned edges). In this situation a collision of two solitons not too far from the edges seems like a many-particle collision since the presence of the edge is equivalent to the presence of the mirror symmetric set of additional solitons according to the obvious rule: $u(-x) = -u(x)$ in the case (25a), and $u(-x) = u(x)$ in the case (25b). In particular, in the former case calculations analogous to those described above yield the following result for the collision of two kinks with velocities $V_1, V_2$ under the condition $1 - V_1^2 < 1 - V_2^2 \ll 1$ (cf. (3)): 

$$\Delta(1 - V_1^2)^{-1/2} = \Delta(1 - V_2^2)^{-1/2}$$ 

$$= \frac{128}{9}e(1 - V_1^2)(1 - V_2^2)^{-1/2} \sinh \delta \sech^2\delta$$ 

$$\times (1 - 2 \sech^2\delta)(1 - 8 \sech^2\delta + 8 \sech^4\delta) \quad (26)$$ 

(cf. (9), (12), (16)). Here $\delta = x_0/(1 - V_2^2)^{1/2}$, and $x_0$ is the $x$-coordinate of the second soliton's center at the collision moment, i.e. $x_0$ characterizes the degree of overlapping between the slower soliton and the edge during the collision. It is clear that (26) satisfies conservation of energy. In the case (25b) the result is zero in the approximation considered.

4. In the above approximation the collision of two kinks described by (1) is purely elastic. However, in reality it is inelastic due to the perturbation-induced energy emission [2]. As in the analogous problem for the conservatively perturbed nonlinear Schrödinger equation [1], the inelasticity of the collision is characterized by the total emitted energy $E$. In the case when a kink and an antikink collide with small velocities $\pm V$ radiative energy losses may result in their binding into a bound state (breather). The threshold velocity for this inelastic process has been found in ref. [2] using numerical data from ref. [6]: 

$$V_{\text{thr}} = \frac{3}{2}\sqrt{E/2} \approx 1.112e.$$ 

In the opposite case, when the colliding kinks are fast i.e. $(1 - V^2)^{1/2} \ll 1, E$ has been calculated purely analytically by the present authors [7]: 

$$E \approx 8.78e^2\{\frac{1}{2}(1 - V^2)\}^3 - \sigma \times 1.10e^2\{\frac{1}{2}(1 - V^2)\}^5,$$ 

$\sigma$ being the relative polarity of the kinks.

The results (27) and (28) can be employed for the investigation of radiative effects accompanying reflection of one kink from the edge (25a) or (25b). Firstly, expression (27) gives the threshold velocity for radiative annihilation of a slow kink on the edge (25a); in the case (25b) the annihilation is not possible since two unipolar kinks cannot annihilate into a bound state [2] (in this connection it is worthwhile to mention that the threshold velocity for radiative capture of a slow kink by the localized inhomogeneity (17) has been found in ref. [2]). Secondly, half the expression (28) gives the energy emitted by a fast kink reverberating from the edge. As is seen from (28), the energy asymptotically (when $1 - V^2 \rightarrow 0$) coincides for both types of the boundary conditions ($\sigma = -1$ corresponds to (25a), and $\sigma = +1$ corresponds to (25b)).

The full variant of the present and preceding [1] papers containing details of calculations and some additional results (e.g., kink-breather and breather-breather many-particle interactions) will be published elsewhere.

The authors are indebted to V.E. Zakharov, A.M. Kosevich, V.A. Marchenko, S.V. Manakov and L.A. Ostrovsky for useful discussions.