NONLINEAR SURFACE ELASTIC MODES IN CRYSTALS

V.I. GORENTSVEIG, Yu.S. KIVSHAR, A.M. KOSEVICH and E.S. SYRKIN
Institute for Low Temperature Physics and Engineering, UkrSSR Academy of Sciences, 47 Lenin Avenue, Kharkov 310164, USSR

Received 29 December 1989; accepted for publication 16 January 1990
Communicated by V.M. Agranovich

The influence of nonlinearity on shear horizontal surface elastic waves in crystals is described on the basis of the effective nonlinear Schrödinger equation. It is shown that the corresponding solutions form a set of surface modes and the simplest mode coincides with the solution proposed by Mozhaev. The higher order modes have internal frequencies caused by the nonlinearity. All these modes decay in the crystal as $u_0 \exp(-z/z_0)$ at $z \gg z_0$ ($u_0$ is the wave amplitude at the surface). The creation of the modes from a localized surface excitation has a threshold. The stability of the modes is discussed.

1. Introduction

Among various types of surface waves (SWs) a special type, the so-called shear horizontal waves, may be distinguished. In the linear (and local) elasticity theory these SWs are absent at both a surface [1] and an interface [2]. But a plane (bulk) shear horizontal wave propagating along a free surface of a homogeneous isotropic half space (or along special directions of crystals) is an exact solution of elasticity theory equations with boundary conditions of elastic stress absence on the surface. However, a bulk wave is known to be "unstable" and may become a surface one under a small change in the parameters of the system. There is a number of physically important situations in which these bulk waves may be localized near the surface. Under the piezoelectric effect in a medium the so-called Gulyaev–Bleustein (surface) waves arise [3,4], these waves were studied by crystal–lattice methods in refs. [5,6]. Taking into account a spatial dispersion (nonlocal elasticity theory) as well as surface distortions (the so-called capillary parameters) also leads to waves localized at a free surface [6,7] or an interface [8,9].

In recent years there has been interest in the influence of nonlinearity on the properties of surface electromagnetic (see, e.g., ref. [10]) and acoustic (see, e.g., ref. [11]) waves. Taking into account nonlinear terms in the corresponding equations of motion leads to new effects, e.g., the generation of the second harmonic wave. These effects are important because for a relatively small source power the oscillation amplitude at the surface may be not very small. Mozhaev has studied a pure shear horizontal wave in the presence of nonlinearity [12] (see also ref. [13]). As was demonstrated in ref. [12], in a nonlinear elastic semi-infinite medium new surface acoustic waves exist, localization of which near the surface is entirely due to nonlinearity. But in these papers the simplest solution of the equations of the nonlinear elasticity theory was given. Our paper aims at presenting nonlinear shear horizontal waves of more general type, which can be considered as a non-stationary extension of the solution obtained in refs. [12,13]. We demonstrate that the nonlinear equations describing shear horizontal waves may be transformed into an effective nonlinear Schrödinger (NLS) equation. The solutions of the equation may be obtained as a set of nonlinear surface modes, the simplest mode is, in fact, the approximate solution obtained in ref. [12]. The higher order modes have internal frequencies generated by nonlinear effects, the frequencies are proportional to the surface amplitude squared. All these modes are localized at the surface, the localization is entirely due to nonlinearity. Using the results of the inverse scattering
transform for the NLS equation, we also study the generation of the modes from an arbitrary pulse localized near the surface and predict threshold effects. We also discuss the stability problem for the solutions and demonstrate that the surface waves are stable against envelope perturbations. In the same case, the carrier plane wave is modulationally unstable and may give rise to formation of surface solitons.

2. Basic equations

Let us consider a semi-infinite crystal with cubic symmetry, whose two axes lie on the surface (see fig. 1). We study the case when a wave propagates along the x-axis, the displacement vector $u$ is directed along the y-axis, and the z-axis of the Cartesian coordinates $x, y, z$ is direct along the normal to the surface $z=0$. The motion equations and the boundary condition of elastic stress absence at a free surface of solids ($z=0$) may be obtained in the framework of the nonlinear elasticity theory for the displacement function $u(x, z; t)$ [14]:

$$\frac{1}{c^2} u_{tt} = u_{xx} + u_{zz} + (\alpha u_x^2 + \beta u_z^2) u_{xx} + (\beta u_z^2 + \alpha u_x^2) u_{zz} + 4\beta u_x u_z u_{xz},$$

(1)

with the nonlinear boundary condition

$$u_x(1 + \beta u_z^2 + \frac{1}{2} \alpha u_x^2)|_{z=0} = 0,$$

(2)

where the subscripts $x, z,$ and $t$ mean the derivations with respect to corresponding variables. Eq. (1) has the following parameters: $c = (C_{44}/\rho)^{1/2}$ is the velocity of the linear transversal wave and

$$\alpha = C_{44}^{-1}(\frac{1}{2} C_{444} + 3 C_{44} + \frac{3}{4} C_{11}),$$

$$\beta = C_{44}^{-1}(\frac{1}{4} C_{4466} + C_{144} + 2 C_{456} + C_{44} + \frac{1}{2} C_{12}),$$

where $C_{ik}, C_{ikh}, C_{ikhm}$ are the crystal modula of second, third, and fourth orders, respectively (in the notation of Voight). To obtain the nonlinear equations (1), (2) one needs to use the expansion of the crystal elastic energy in the displacements with accuracy of terms up to fourth order, because for pure shear horizontal waves the first nonvanishing terms in the equations of motion are those of third order. It is valid for an isotropic medium and a cubic crystal.

In the linear limit the solution of eqs. (1), (2) does not depend on the z-coordinate (i.e., it is no SW) and has the form

$$u = u_0 \exp(ikx - i\omega_0 t),$$

where the wave number $k$ and the wave frequency $\omega_0$ are connected by the dispersion relation $\omega_0 = c^2 k^2$.

Let us consider the solution of eqs. (1), (2) in the nonlinear case in the form

$$u = U(x, z; t) \exp(ikx - i\omega t) + c.c.,$$

(3)

where $U$ is an envelope slowly varying in comparison with the fast oscillations. Substituting eq. (3) into eq. (1) and omitting (after averaging) oscillating terms which arise due to interaction of the fundamental frequency wave with higher harmonics, we get an equation for the envelope $U$ in the form

$$\frac{1}{c^2} U_{tt} - 2i[(k/c) U_t + k U_x] - U_{xx} = U_{zz} + F(U),$$

(4)

where the term $F(U)$ describes the nonlinear terms,

$$F(U) = \alpha(-k^4 |U|^2 U + U_x^2 U_z^* + 2 |U_x|^2 U_{zz}^*) + \beta k^2 (3U_z^2 U_z^* - 2 |U_z|^2 U_x^2 U_{zz}^* + 2 |U|^2 U_{zz}^*),$$

(5)

(here an asterisk means complex conjugation).

Analogously, similar transformation of the boundary conditions (2) leads to the conditions for the slowly varying envelope,
According to the method, eq. (11) is connected with the linear scattering problem, the so-called Zakharov–Shabat scattering problem, for the auxiliary function \( \Psi (\zeta; \lambda) = (\Psi_1, \Psi_2) \):

\[
\frac{\partial}{\partial \zeta} \Psi_1 = i\lambda \Psi_1 + iv(\zeta, 0) \Psi_2 ,
\]

\[
\frac{\partial}{\partial \zeta} \Psi_2 = -i\lambda \Psi_2 + iv^*(\zeta, 0) \Psi_1 ,
\]

with \( v(\zeta, 0) \) (which falls off fast enough at \( \zeta \to \pm \infty \)) being the initial condition for eq. (11). Each discrete eigenvalue \( \lambda = \xi + iv \) corresponds to a soliton solution with amplitude \( 2\eta \) and velocity \( -4\xi \). The important information about the soliton solutions is contained in the so-called Jost coefficients, \( a(\lambda) \) and \( b(\lambda) \), i.e. the scattering amplitudes related to the spectral problem (13). In particular, the zeroes of \( a(\lambda) \) are the discrete eigenvalues of the problem (13). According to the IST, there are exact solutions of the NLS equation (11) describing collisions of \( N \) solitons. These \( N \)-soliton solutions are characterized by a set of complex variables \( \{\lambda_j, C_j\} \), \( j = 1, 2, \ldots, N \). The complex \( \lambda_j \) are the “poles” of the soliton solution, and the \( C_j \) are its “residues”. Zakharov and Shabat [18] showed that bound states are obtained only if all the poles \( \lambda_j \) are imaginary ones. If they are not, the solution separates into different soliton pulses [16–18].

We will apply the results of the IST to construct solutions of eq. (11) in the problem with boundary conditions (12).

4. A set of nonlinear surface modes

To satisfy the boundary conditions (12) and obtain the SW solutions, we will use the so-called \( N \)-soliton solutions of eq. (11); the general \( N \)-soliton solution may be obtained from the relations

\[
v(\zeta, \tau) = -2 \sum_{k=1}^{N} g_k^* \Psi_{2k}^* ,
\]

where

\[
g_k \equiv g_k(\zeta, \tau) = C_k(0) \exp(i\lambda_k \zeta - i2k \tau) ,
\]

and \( \Psi_{2k} \) is defined by the equations

\[
\frac{\partial}{\partial \zeta} \Psi_{2k} = -i\lambda_k \Psi_{2k} + iv^*(\zeta, 0) \Psi_{1k} ,
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\[
\frac{\partial}{\partial \zeta} \Psi_{2k} = -i\lambda_k \Psi_{2k} + iv^*(\zeta, 0) \Psi_{1k} ,
\]
\[ \psi_j + \sum_{k=1}^{N} \frac{g_k g_k^*}{\lambda_j - \lambda_k^*} \psi_{2k} = 0, \]

\[ \psi_{2k} - \sum_{k=1}^{N} \frac{g_k g_k^*}{\lambda_j^* - \lambda_k^*} \psi_{1k} = g_j^* \]

(16)

(see for details ref. [18]). For example, in the case \( N=1 \) the IST formulas (14)–(16) lead to the well-known one-soliton solution

\[ v(\zeta, \tau) = 2\eta \frac{\exp[-2i\zeta^2 - 2i(\zeta^2 - \eta^2)\tau - i\phi^{(0)}]}{\cosh[2\eta(\zeta - \zeta^{(0)} + 2\zeta\tau)]}, \]

(17)

where \( \phi^{(0)} = \arg C_1(0) \) and \( \zeta^{(0)} = (2\eta)^{-1} \times \ln|C_1(0)|/2\eta \). To satisfy the boundary condition (12a), we will consider symmetric solutions \( v(\zeta, \tau) = v(-\zeta, \tau) \). The set of solutions includes SWs for which we have to put \( \zeta^{(0)} = 0 \) and \( \zeta = 0 \) (\( j = 1, 2, ..., N \)) in eqs. (14)–(16). The simplest solution of such a type is the simple surface wave stipulated by the one-soliton solution (17) at \( \eta =\frac{1}{2} \),

\[ v(\zeta, \tau) = \frac{\exp(\tau/2)}{\cosh \zeta}. \]

Taking into account the relations between the dimensionless variables and the time and the space coordinate, we may present the approximate solution of eqs. (1), (2) as

\[ u(z, t) = U_0 \sech(z/z_0) \exp(ikx - i\omega t) + \text{c.c.}, \]

(19)

where the resulting wave frequency

\[ \omega = kc - \frac{1}{2T} = kc - \frac{1}{4} |\alpha|ck^3U_0^2 \]

(20)

is connected with the wave number \( k \) and the wave amplitude \( U_0 \). The solution (19), (20) is exactly the same as that obtained by Mozhaev [12] using the simple substitution. Indeed, for the small amplitude \( U_0 \) we have the dispersion relation

\[ \frac{\omega^2}{c^2} = (k - \frac{1}{4} |\alpha|k^3U_0^2)^2 \approx k^2 - \frac{1}{4} |\alpha|k^4U_0^2, \]

(21)

But for a simple renormalization (\( \alpha \to \frac{1}{4} \alpha \)) this coincides with the result of ref. [12].

It is interesting to note that the solution (19), (20) is localized near the surface (instead of the linear wave) on a length of order of \( z_0 \sim U_0^{-1} \), i.e. the SW is generated by the nonlinearity.

The higher order soliton solutions have more parameters, one of them being the surface amplitude. Therefore, the \( N \)-soliton solution of the NLS equation gives rise to a set of surface waves, the so-called nonlinear modes. To obtain symmetric solutions of such a type, in eqs. (14)–(16) we need to put

\[ C_j^2 = \prod_{k=1}^{N} (\eta_j + \eta_k) \prod_{k=1}^{N} |\eta_k - \eta_j|, \]

where \( \eta_j \) and \( C_j \equiv C_j(0) \) are the scattering data and \( \lambda_j \equiv i\eta_j \). The symmetric soliton of order \( N = 2 \) can be written down explicitly in the general case (see ref. [19]),

\[ v(\zeta, \tau) = \frac{4\eta_1(\eta_1 + \eta_2)}{|\eta_2 - \eta_1|} \exp(2i\eta_1^2\tau) \frac{A(\zeta, \tau)B(\zeta, \tau)}{A(\zeta, \tau)B(\zeta, \tau)}, \]

(22)

where

\[ A(\zeta, \tau) = \cosh(2\eta_2 \zeta) \]

\[ + \frac{\eta_2}{\eta_1} \cosh(2\eta_1 \zeta) \exp(i\Omega \tau), \]

(23)

\[ B(\zeta, \tau) = \cosh(2(\eta_1 + \eta_2) \zeta) \]

\[ + \left( \frac{\eta_1 + \eta_2}{\eta_2 - \eta_1} \right)^2 \cosh(2(\eta_2 - \eta_1) \zeta) \]

\[ + \frac{4\eta_1 \eta_2}{(\eta_1 - \eta_2)^2} \cos(\Omega \tau), \]

(24)

\[ \Omega = 2(\eta_1^2 - \eta_1). \]

(25)

The solution (22)–(25) has two independent parameters \( \eta_1 \) and \( \eta_2 \). One of them must be fixed because we have introduced the amplitude of the wave at the surface. To determine the values, we use the following notations,

\[ \eta_1 + \eta_2 = \frac{1}{2}, \quad \eta_2 - \eta_1 = -\Omega. \]

(26)

Then the parameters \( \Omega \) and \( U_0 \) describe the mode of second order. In these notations

\[ \eta_2 = \frac{1}{2}(\frac{1}{2} + \Omega), \quad \eta_1 = \frac{1}{2}(\frac{1}{2} - \Omega), \]

(27)

and the solution amplitude changes between the values
\[ |\nu(0, (\pi n / \Omega))| = 2(\eta_2 - \eta_1) = 2\Omega \]

and

\[ |\nu(0, (\pi / \Omega)(\frac{1}{2} + n))| = \frac{2(\eta_2^3 - \eta_1^3)}{\sqrt{\eta_1^2 + \eta_2^2}} = \frac{2\sqrt{2}\Omega}{\sqrt{1 + 4\Omega^2}}. \]

The solution has the period \( T_0 = \pi / (\eta_2^2 - \eta_1^2) = 2\pi / \Omega \), and describes oscillations near the surface. Fig. 2 shows the sequence of \( N = 2 \) solutions (22)–(24) of the varying ratio \( \eta_1 / \eta_2 = (\frac{1}{2} - \Omega) / (\frac{1}{2} + \Omega) \). In the physical variables (10) the frequency of these internal oscillations is

\[ \Omega_{\text{non}} = \frac{\Omega}{T} = \frac{1}{2} |\alpha| c k^3 \Omega U_0^2, \quad (28) \]

and is in fact stipulated by nonlinearity.

Using the higher order solitons of the NLS equation, we may present the form of the higher-order nonlinear surface modes corresponding to the higher order solitons, the \( N \)th mode has \( N \) free parameters,
one of them being the surface amplitude.

The N-soliton solutions (14)–(16) are partial solutions of the NLS equation (11). The IST allows us to describe the formation of the higher order modes from a localized initial wave pulse \( \nu(\zeta, \tau = 0) \). The problem may be easily solved in the case

\[
\nu(\zeta, \tau = 0) = \tilde{\nu}(\zeta) \exp(i\gamma), \tag{29}
\]

where \( \tilde{\nu}(\zeta) \) is a real function and \( \gamma \) is a constant phase (see ref. [20]). The same results may be obtained for the symmetric initial pulse \( \nu(-\zeta) = \nu(\zeta) \) allowing the boundary conditions (12). The main point of this study is the investigation of the zeroes of the scattering coefficient \( a(\lambda) \) lying on the imaginary axis. The results are as follows. The arbitrary initial excitation (29) will be transformed into a surface mode provided the threshold condition [20] is fulfilled,

\[
S = \int_{-\infty}^{\infty} \tilde{\nu}(\zeta) \, d\zeta \geq \frac{1}{2} \pi .
\]

Such an initial wave leads to the corresponding mode of order \( N \), the number of the mode being defined as the number of solitons [20], \( N = \lfloor \frac{1}{2} + \pi^{-1} S \rfloor \), where \( \lfloor \rfloor \) means integer part. Therefore, the surface excitation (29) may form only one mode from the set of localized modes, its number (order) is defined by the area under the curve \( \tilde{\nu}(\zeta) \).

5. Stability of the solutions

The surface waves are obtained as solutions of the nonlinear Schrödinger equation (11), (12) with boundary condition of elastic stress absence at the free surface of solids \( (u_z = 0) \). In a number of cases, the boundary condition is modified (see, e.g., refs. [13,14]) and the surface waves have to be stable against those modifications. The problem may be easily considered for the case of the so-called capillary parameters at the boundary [7,8,13], when the boundary condition is modified to the form

\[
u_z|_{z=0} = (g + h_{66})u_{xx}|_{z=0} - \rho_s u_{tt}|_{z=0}, \tag{30}
\]

where \( g, h_{66} \) and \( \rho_s \) are the capillary parameters. A similar condition can be obtained for the envelope \( U \):

\[
U_z|_{z=0} = \gamma U, \quad \gamma = \rho c^2 - (g + h_{66}) k^2, \tag{31}
\]

The one-soliton solution in the presence of the condition (31) may be taken in the slightly modified form

\[
U = U_0 \operatorname{sech}[(z-Z)/z_0] \exp(ikx-i\omega t) + \text{c.c.}
\]

[cf. eq. (19)] with the dispersion relation (20) and the equation for the parameter \( Z \):

\[
\tanh(Z/z_0) = \gamma z_0 \tag{32}
\]

According to eq. (32), the maximum of the wave amplitude \( U_0 \) is displaced from the surface and the surface amplitude becomes smaller: \( U_0/(\gamma z_0)^{1/2} \). Such a wave may exist provided \( \gamma z_0 < 1 \). Similar results were obtained in ref. [13].

To investigate the stability of the higher order modes we present eq. (11) with the boundary condition \( \nu_z|_{z=0} = (\gamma z_0)\nu(0) \) in the following equivalent form,

\[
i\nu_t + \frac{1}{2} \nu_{xx} + |\nu|^2\nu = \gamma z_0 \delta(\zeta) \nu, \quad \nu(-\zeta) = \nu(\zeta),
\]

which allows us to use directly the perturbation theory for solitons of the NLS equation [2]. Cumberstone but simple calculations for the symmetric two-soliton solution demonstrate the stability of such solutions against changing the boundary condition. The stability manifests itself in the results: \( \delta\zeta = 0 \) \( (j=1,2) \) (see eqs. (14)–(17) of the IST approach). But for arbitrary values of \( \gamma \) we cannot present simple analytical formulas as in the case of the one-soliton mode.

As was demonstrated in section, 3, the exact equation for the wave amplitude envelope has the form (4), i.e. it is in fact the two-dimensional NLS equation. The main problem arising in such a case is the stability of one-dimensional solutions against transversal perturbations (see, e.g., refs. [17,22]). As is well known, one-dimensional soliton solutions of the NLS equation are unstable for \( |z| < \infty \). Indeed, let us consider the stability problem for the two-dimensional NLS equation which follows from eq. (4) under the conditions \( U_{tt} \ll kc U_z, \ U_z \ll kU \):

\[
i\nu_t + \frac{1}{2} \nu_{xx} + |\nu|^2\nu = -\frac{1}{2} \epsilon^2 \nu_{\xi \xi}, \tag{33}
\]

where we have used the variables \( \tau, \zeta \) and \( \xi = \epsilon x'/z_0 \), \( \epsilon \) being a small parameter. The term in the r.h.s. of eq. (33) may be considered as a small perturbation.
To study the stability problem for the one-soliton solution at $|z| < \infty$, we will find the perturbed solution in the form

$$
\nu(\zeta, \tau, \xi) = \frac{\exp(i [\tau - \sigma(\tau, \xi)])}{\cosh[\zeta + \theta(\tau, \xi)]},
$$

(34)

where $\sigma$ and $\theta$ depend on the "slow" time, $\sigma, \sim \epsilon \sigma$, $\theta, \sim \epsilon \theta$ and the transversal coordinate $\xi$. Direct application of the perturbation theory in the framework of the so-called adiabatic approximation [21] allows us to obtain the equations for the parameters $\sigma(\tau, \xi)$ and $\theta(\tau, \xi)$ (see ref. [22], and also ref. [17]):

$$
\sigma_{\tau\tau} + \epsilon^2 \sigma_{\xi\xi} = 0,
$$

(35a)

$$
\theta_{\tau\tau} - \frac{1}{2} \epsilon^2 \theta_{\xi\xi} = 0.
$$

(35b)

Eq. (35b) leads to an instability of the one-dimensional NLS soliton in the two-dimensional system. The same approach may be applied to the problem under consideration. The main (and very important) difference is related to the boundary condition (12a). The boundary condition fixes the position of the solution (31), i.e. $\theta = 0$. As a result, the envelope of the surface wave (18) is stable against transversal perturbations. We believe that an analogous result is valid for the higher order modes, too.

Another very important question arising in nonlinear problems is the modulation instability of the cw solutions. As is well known, in a one-dimensional nonlinear system the monochromatic plane wave, the so-called cw solution,

$$
u = u_0 \exp[ikx - i\omega(k, u_0)]
$$

(36)

is stable provided the Lighthill condition is valid,

$$
\frac{\partial \omega}{\partial \omega} \frac{\partial \omega}{\partial u_0^2} > 0,
$$

(37)

In the other case, the cw solution is unstable in the formation of sidebands (the so-called Benjamin–Feir instability). Such a modulation instability usually corresponds to creation of solitons in nonlinear systems.

In our problem, the frequency of the resulting solution is corrected by the nonlinear term which is proportional to the surface amplitude squared (see eqs. (19), (20)). It is easy to see that for the SW (19) the condition (37) is not valid. It means that the carrier plane wave along the propagation direction ($x$-coordinate) is modulationally unstable and may form, for example, a succession of surface envelope solitons (similar to Rayleigh waves [23]). In this case the resulting solution describing a surface wave in our system (1), (2) will be localized not only in the $z$-direction but also along the surface, i.e., in the $x$-direction. Such solutions have to be two-dimensional solitons. We have now no analytical approach to describe the two-dimensional waves, but we believe that the $z$-dependence of such solutions will be similar to that investigated in the present paper.

At last, we will shortly discuss the influence of a dispersion of the bulk equation (1) on the presented results. Taking into account such a dispersion as additional terms $\gamma_1 u_{xxxx} + \gamma_2 u_{xxxxx} + \gamma_3 u_{xxxxxx}$ in the r.h.s. of eq. (1) leads to the linear dispersion relation $\omega^2(k) = c^2 k^2 (1 - \gamma_1 k^2)$. The nonlinear waves may be constructed in this case according to a similar approach as envelopes of the linear waves, $-\exp(ikx - ikc_0/\sqrt{1 - \gamma_1 k^2})$ (cf. eq. (3)). In particular, the equation for the wave envelope $U$ has the form (8) with the substitution $U_{\xi\xi} - (1 - \gamma_1 k^2) U_{\xi\xi}$, in the reference frame moving with the group velocity $v_g = (c^2 k/\omega_0) (1 - 2\gamma_1 k^2)$ along the $x$-direction. It is easy to see that the influence of the dispersion terms in the typical case $\gamma_1 k^2, \gamma_1 k > 1$ is almost negligible and the nonlinear modes presented above have the same structure and dynamics.

6. Conclusions

We have demonstrated that shear horizontal waves may be localized near a surface due to nonlinearity and there is a set of these waves in the form of nonlinear modes in crystals. The penetration length of the waves is determined by eq. (9) to be $kU_0 \sim 10^{-3}$--$10^{-4}$ and $|\alpha| \sim 10^3$ (see ref. [15]) yields $z_0 \approx (7-70) \lambda_0$, $\lambda_0$ being the wave length. The higher order modes have internal frequencies related to their amplitudes and correspond to the so-called $N$-soliton solutions of the effective nonlinear Schrödinger equation. The modes' envelopes are stable in the presence of boundary conditions of different types, but an instability of the carrier wave propagating along the crystal surface in the presence of nonlinearity may lead to creation of surface solitons local-
ized in the propagation direction in the crystal.

Acknowledgement

We would like to thank Dr. Yu.A. Kosevich, Dr. A.S. Kovalev, Professor G. Maugin and Dr. V.G. Mozhaev for useful discussions.

References