A potential of incoherent attraction between multidimensional solitons

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Abstract

We obtain analytical expressions for an effective potential of interaction between two- and three-dimensional (2D and 3D) solitons (including the case of 2D vortex solitons) belonging to two different modes which are incoherently coupled by cross-phase modulation. The derivation is based on calculation of the interaction term in the full Hamiltonian of the system. An essential peculiarity is that, in the 3D case, as well as in the case of 2D solitons with unequal masses, the main contribution to the interaction potential originates from a vicinity of one or both solitons, similarly to what was recently found in the 2D and 3D single-mode systems, while in the case of identical 2D solitons, the dominating area covers all the space between the solitons. Unlike the single-mode systems, stable bound states of mutually orbiting solitons are possible in the bimodal system. © 1999 Published by Elsevier Science B.V.

1. Introduction

Interaction between self-focusing cylindrical beams (spatial solitons) in bulk nonlinear media is a problem of obvious interest both by itself and for applications. This interaction was studied experimentally and by means of numerical simulations in photorefractive media [1,2], and was simulated in detail in an isotropic model with the second-harmonic-generating (SHG) nonlinearity [3]. In the latter model, it was demonstrated that the spiraling bound state of two cylindrical beams is unstable. A general analytical expression for a potential of interaction between far separated two- and three-dimensional (2D and 3D) solitons was very recently derived in Ref. [4]. The potential also predicts an instability of the orbiting bound state of two solitons.

A very convenient model for the study of the multidimensional solitons and their interactions is the cubic–quintic nonlinear Schrödinger (CQNLS) equation, in which the cubic nonlinearity is self-focusing, giving rise to the beams (2D solitons) or “light bullets” [5] (3D solitons), while the quintic term is self-defocusing, precluding the wave collapse in the model,

\[ iu_t + \nabla^2 u + |u|^2 u - g|u|^4 u = 0. \] (1)

The coefficients in (1), except for \( g \), can be set equal to 1 by means of scale transformations, whereas \( g \) is the quintic-to-cubic nonlinear susceptibilities ratio. In the application to nonlinear optical media, the temporal variable \( t \) in (1) must be replaced by the propaga-
tion distance \( z \), while the role of the third transverse variable is played by the “reduced time”, \( t - z/V_g \), \( V_g \) being the mean group velocity of the carrier wave (this implies that the temporal dispersion in the medium is anomalous) [5]. Finally, the Hamiltonian of this model is

\[
H_u = \int \left( \|\nabla u\|^2 - \frac{1}{2} |u|^4 + \frac{1}{3} g |u|^6 \right) \, \text{d}r.
\]  

(2)

Vortex beams, with the vorticity (“spin”) \( s = 1 \), and interactions between them, described by Eq. (1), were simulated by Quiroga-Teixeiro and Michinel [6]. A remarkable result is the numerically discovered robustness of the vortex beams (which were found to be strongly unstable against azimuthal perturbations in the SHG model [7]). Note that the model (1) is not merely the simplest one that gives rise to stable multidimensional solitons: according to experimental data [8], the combination of the focusing cubic and defocusing quintic terms adequately represents the nonlinear optical properties of some real materials.

The effective potential of the intersoliton interaction derived in Ref. [4] applies to a wide class of models, including Eq. (1). However, it does not apply to bimodal systems including two equations with incoherent nonlinear coupling between them (mediated by the cross-phase modulation in nonlinear optical media), in the case when the two solitons (beams) belong to different modes. The simplest bimodal generalization of the model based on the Hamiltonian (2) is furnished by the Hamiltonian \( H = H_u + H_v + H_{\text{int}} \), where \( H_i \) is the same expression as (2) with the field \( u \) substituted by another field \( v \), and the interaction part of the Hamiltonian is

\[
H_{\text{int}} = \int \left[ -\beta |u^2|v|^2 + \alpha (|u|^4|v|^2 + |u|^2|v|^4) \right] \, \text{d}r.
\]

(3)

with, generally speaking, arbitrary positive constants \( \alpha \) and \( \beta \). The full Hamiltonian of the bimodal system gives rise to the equations

\[
iu_t + \nabla^2 u + (|u|^2 + \beta |v|^2)u
- (g|u|^4 + 2\alpha |u^2|^2|v|^2 + \alpha |v|^4)u = 0,
\]

(4)

\[
iv_t + \nabla^2 v + (|v|^2 + \beta |u|^2)v
- (g|v|^4 + 2\alpha |u|^2|v|^2 + \alpha |u|^4)v = 0.
\]

(5)

Commonly known examples of optical bimodal systems are provided by two orthogonal polarizations of light, or two light waves with different carrier wavelengths [9]. In the latter case, as well as in the former one with the circular polarizations, the cubic cross-phase-modulation coefficient is \( \beta = 2 \). In the case of two linear polarizations, \( \beta = \frac{2}{3} \) (and the usual assumption is to drop additional four-wave mixing terms). The constant \( \alpha \) is left here to be arbitrary, but, in the most interesting cases, the second term in (3) will only produce a small correction to the effective interaction potential.

As well as in the case of the single-mode system, the interaction between solitons in different modes depends on the separation between them, but, unlike the single-mode case, it is not sensitive to a phase difference between the solitons, hence the interaction is expected to be simpler than inside the same mode. The objective of this work is to find an effective potential of the interaction between 2D and 3D solitons in the bimodal system, including the cases when the interacting solitons are both identical and different (the interaction between 2D solitons with different vorticities is also included). The interaction between identical 2D beams was recently considered in Ref. [10], but using an artificially introduced Gaussian ansatz for the soliton. As well as it was done in Ref. [4] for the single-mode system, in this work we find the interaction potential in a general analytical form. However, the derivation is essentially different from that developed in Ref. [4]; in particular, the derivation proves to be very different for the 2D and 3D cases, while in the single-mode system these two cases were very similar. In Section 2, we derive the potential for the interaction between 2D solitons (spatial beams) with unequal masses. We explicitly consider two limit cases, when the masses of the interacting solitons are very different or nearly equal. The latter case demonstrates a singularity in the limit of equal masses, therefore the interaction between identical 2D solitons should be considered separately, which is done in Section 3. The result, and the way to obtain it, turns out to be drastically different from the case of unequal masses: when the masses are not equal, a dominating contribution to the effective interaction potential is produced by a vicinity of the soliton having a larger mass (which is similar to the situation for the single-mode system [4]), while, in the equal-mass case, a domi-
nating area is located *between* the solitons, in contrast with the case of the single-mode system. In Section 4, the potential is derived for the 3D solitons with equal masses. In this case, the derivation is similar to that for the 2D solitons with the unequal masses, but it gives rise to an additional logarithmic factor.

The results are summarized in Section 5, where we conclude, in particular, that the obtained potentials give rise to two bound states of the solitons orbiting around each other, one of which is stable (which was already concluded in Ref. [10]), on the contrary to the orbiting states in the single-mode models [4] (including the SHG one [3]), which are all unstable. This difference, which is, obviously, very important for applications, is due to the fact that, in the bimodal system, the interaction potential does not depend on the phase difference between the two solitons.

2. The interaction between different two-dimensional solitons

A general 2D stationary solution to Eq. (4), with an internal frequency \( \omega = -\mu^2 \), is looked for in the form

\[
u_s = \exp(i\mu^2 t + is\theta)U(r),
\]

where the \( s \) is an integer spin (vorticity), and a real function \( U(r) \) satisfies the equation \( U'' + r^{-1}U' - s^2r^{-2}U + U^3 - gU^5 = \mu^2U \), which can be easily solved numerically [6]. A soliton solution is defined by its asymptotic form at \( r \to \infty \),

\[
U(r) \approx A_s(\mu) r^{-1/2} \exp(-\mu r),
\]

where the amplitude \( A_s(\mu) \) is to be found numerically. We will consider a situation with the solitons of the form (6), (7) in each mode \( u \) and \( v \), that have, generally, different spins \( s_1 \) and \( s_2 \) and different frequency parameters \( \mu_1 \) and \( \mu_2 \) which determine effective masses of the solitons. A size of the soliton can be estimated, pursuant to Eq. (7), as \( \mu^{-1} \). We will consider the case when the separation between the solitons is much larger than their proper sizes, i.e., \( R\mu_{1,2} \gg 1 \).

The interaction Hamiltonian (3) allows one to define an effective interaction potential for two separated solitons, approximating the two-soliton configuration by a linear superposition of two isolated solitons, and substituting it into (3) [11]. This approach requires actual calculation of the integrals in (3), which can be done for 2D solitons in an exact form only in exceptional cases (see, e.g., Ref. [12]), another drawback being that a distortion of the "tail" of each soliton due to its interaction with the "body" of the other one is ignored. In the work [10], the necessary integral was evaluated, assuming a Gaussian ansatz for the isolated solitons. However, the corresponding effective interaction potential, decaying \( \sim \exp(-\text{const} \times R^2) \), was actually produced by the ansatz rather than by the model. In fact, the potential must decay as \( \exp(-\text{const} \times R) \), see below.

In the work [4], another approach to the calculation of the effective potential was developed for the single-mode systems, following the way elaborated earlier for 1D solitons in Ref. [13]. This method did not require knowing the internal structure of the soliton, and did not imply neglecting the distortion of each soliton's tail due to its overlapping with the other soliton. All that is necessary to know about the individual solitons for the application of this method, is only the asymptotic amplitudes \( A_s(\mu) \) in (7). Here, we will apply a similar method to the bimodal system (4), (5), although technical details will be essentially different from those in the case of the single-mode systems.

We start, still assuming the linear superposition of the two solitons \( u_{s_1} \) and \( v_{s_2} \) (see Eqs. (6) and (7)) with widely separated centers, and setting, for the definiteness, \( \mu_1 > \mu_2 \). Because of the exponential decay of the fields, it is obvious that a dominant contribution to the interaction potential (3) will be produced by a vicinity of the soliton with \( \mu = \mu_1 \).

First, we consider the case \( \mu_2 \ll \mu_1 \), i.e., a light soliton (beam) interacting with a heavy one. Substituting the field \( v \) for the light soliton by the asymptotic expression (7), and neglecting its small variation over the size of the narrow heavy soliton, we can easily cast the expression for the interaction potential into the form

\[
H_{\text{int}} \approx A_{s_2}^2(\mu_2) R^{-1} \exp(-2\mu_2 R) \\
\times \left[ -\beta \int |u_{s_1}(r; \mu_1)|^2 \, dr + \alpha \int |u_{s_1}(r; \mu_1)|^4 \, dr \right] \\
\equiv (-\beta m_1 + \alpha \tilde{m}_1) A_{s_2}^2(\mu_2) R^{-1} \exp(-2\mu_2 R),
\]

(8)
where $m_1 \equiv \int |u_1(r; \mu_1)|^2 \, dr$ and $\tilde{m}_1 \equiv \int |u_1(r; \mu_1)|^4 \, dr$ are two integral characteristics of the heavy soliton, $m_1$ being, in fact, its effective mass. Thus, both attraction and repulsion between the light and heavy solitons may take place, depending on the sign in front of expression (8).

Another interesting case is $\mu_1 - \mu_2 \equiv \Delta \mu \ll \mu_2 \equiv \mu$ (i.e., the interaction between nearly identical solitons, provided that $s_1 = s_2$). In this case, following Ref. [4], we assume that, in terms of the polar coordinates $(r, \theta)$ with the origin at the center of the heavier (first) soliton, a main contribution to $H_{\text{int}}$ comes from the distances $\mu^{-1} \ll r \ll R$, where both solitons may be approximated by the asymptotic expressions (7). Then, it is straightforward to obtain the following expression corresponding to the first term in (3) (cf. the corresponding expressions and Fig. 1 in Ref. [4]),

$$U_{\text{int}} \approx -\beta A_{\Delta \mu}^4(\mu) A_{s_2}^2(\mu) R^{-1} \int_0^\infty \frac{r}{dr} \int_0^{2\pi} d\theta \, r^{-1} \exp[-2(\mu + \Delta \mu) r] - 2\mu \sqrt{(R + r \cos \theta)^2 + r^2 \sin^2 \theta}].$$

(9)

Here, the small difference $\Delta \mu$, and the difference between $R$ and the exact distance from an integration point $(r, \theta)$ to the center of the second (lighter) soliton, $\sqrt{(R + r \cos \theta)^2 + r^2 \sin^2 \theta}$, are neglected everywhere, except for the argument of the exponential function. Making use of the expansion

$$\sqrt{(R + r \cos \theta)^2 + r^2 \sin^2 \theta} = R + r \cos \theta + \ldots,$$

(10)

and of the formula $\int_0^{2\pi} \exp(-2\mu r \cos \theta) \, d\theta \approx \sqrt{\pi/\mu} \exp(2\mu r)$, valid for $\mu r \gg 1$, we can simplify the integral (9) to the form

$$H_{\text{int}} \approx -\beta A_{\Delta \mu}^4(\mu) A_{s_2}^2(\mu) \sqrt{\pi/\mu} R^{-1} \int_0^\infty \frac{r}{r^{-1/2}} \exp[-2(\Delta \mu) r] \, dr \equiv -\pi \beta (2\mu \Delta \mu)^{-1/2} A_{s_1}^2(\mu) A_{s_2}^2(\mu) R^{-1} \times \exp(-2\mu R).$$

(11)

For small $\Delta \mu$, the integral in (11) is dominated by a contribution from the region $r \sim 1/\Delta \mu \gg \mu^{-1}$, which justifies the use of the asymptotic approximation (7) for the field $u(r)$. A contribution from the second term in the full expression (3), evaluated in the same approximation, demonstrates the same dependence on the separation $R$, but without the large multiplier $(\Delta \mu)^{-1/2}$, therefore this is only a small correction to (11).

### 3. Attraction between identical two-dimensional solitons

Expression (11) diverges in the most interesting case $\Delta \mu = 0$, which corresponds to the interaction between identical solitons (provided that $s_1 = s_2 \equiv s$). The divergence suggests that a region dominating the interaction potential is not that around the soliton, as was the case both in the previous section for the case of $\Delta \mu \neq 0$, and, for the identical solitons, in the single-mode system [4], but a wider region between the two solitons. To calculate $U_{\text{int}}$ in this case, we use the Cartesian coordinates $(x, y)$ defined so that the centers of the two solitons are placed at the points $(\pm 1/2 R, 0)$. Then, using once again the asymptotic expressions (7) for both $u$- and $v$-solitons with $\mu_1 = \mu_2 \equiv \mu$, the interaction potential corresponding to the first term in Eq. (3) is given, after obvious transformations, by

$$H_{\text{int}} = -\beta A_{\Delta \mu}^4 \int d\xi \, d\eta \left[ (\xi^2 + \eta^2 + \frac{1}{2})^2 - 2\eta^2 \right]^{-1/2} \times \exp[-2\mu R \sqrt{(\xi + \eta^2 + \frac{1}{2})^2 + \eta^2\pm \frac{1}{2} R}] + \sqrt{(\xi - \frac{1}{2})^2 + \eta^2}]$$

(12)

where $\xi \equiv x/R$, and $\eta \equiv y/R$. Because the parameter $\mu R$ is large according to the underlying assumption, the integral (12) is dominated by a contribution from a vicinity of points where the argument of the exponential function has a maximum. An elementary analysis shows that the maximum is attained not at isolated points, but rather at the whole segment $|\xi| < \frac{1}{2}$, $\eta = 0$. Expanding the integral in small $\eta^2$ in a vicinity of this segment, we approximate Eq. (12) by an integral that can be easily calculated.
\[ H_{\text{int}} = -\beta A_s \int_{-1/2}^{+1/2} d\xi \left( \frac{1}{4} - \xi^2 \right)^{-1} \]
\[ \times \int_{-\infty}^{+\infty} d\eta \exp \left[ -2\mu R \left( 1 + \frac{1}{2} \frac{\eta^2}{1/4 - \xi^2} \right) \right] \]
\[ = -\beta A_s \sqrt{\frac{\pi^3}{\mu R}} \exp(-2\mu R). \quad (13) \]

Comparing this result with that (11) for the solitons with different masses, we conclude that the divergence in the latter expression at \( \Delta \mu \to 0 \) implies replacement of the pre-exponential factor \( R^{-1} \) by a larger one, \( R^{-1/2} \). Evaluating the second term in (3) in the same approximation, we conclude that it yields an expression differing from (13) just by the factor \( R^{-1} \) instead of \( R^{-1/2} \), i.e., a small correction to (13). Note that, unlike the interaction between heavy and light solitons, which may have either sign, the nearly identical or identical solitons always attract each other, cf. Eqs. (8), (11), and (13).

4. Attraction between three-dimensional solitons

In the 3D case, we consider only the solitons without the internal "spin", i.e., with \( s = 0 \). The 3D soliton has the form \( u = \exp(\mu r^2) a(r) \), with the asymptotic form \( a(r) \approx Ar^{-1} \exp(-\mu r) \) at \( r \to \infty \), cf. Eqs. (6) and (7). Substitution of this asymptotic expression for the fields \( u \) and \( v \) into the integral in the first term of Eq. (3) around each soliton (cf. Eq. (9)) and using the expansion (10) yield

\[ H_{\text{int}} = -4\pi \beta A^4 R^{-2} \int_0^\infty r^2 dr \int_0^\pi \sin \theta d\theta r^{-2} \]
\[ \times \exp\left[ -2\mu r - 2\mu(R + r \cos \theta) \right], \]

where an extra factor 2 takes into regard the fact that, in the case of identical solitons, one has equal contributions from the vicinity of both solitons. After elementary integration over \( \theta \), we arrive at an expression containing a formal logarithmic singularity,

\[ H_{\text{int}} = -2\pi \beta A^4 \mu^{-1} R^{-2} \exp(-2\mu R) \int_0^\infty r^{-1} dr. \quad (14) \]

Actually, the lower and upper limits of the integration are, respectively, \( \sim \mu^{-1} \) and \( \sim R \), so that, with the logarithmic accuracy, the final expression for the effective interaction potential in the 3D case becomes (cf. (13))

\[ H_{\text{int}} = -2\pi \beta A^4 \mu^{-1} R^{-2} \exp(-2\mu R) \ln(\mu R). \quad (15) \]

As for the contribution from the second term in (3), it has the same dependence on \( R \) as (15), but without the large logarithmic factor, so that in this case too, it is a small correction only.

The above analysis was done for identical solitons. If the solitons have a small mass difference, corresponding to a small difference \( \Delta \mu \), the interaction potential is given by essentially the same expression (15), except for a factor of 2, which is absent if \( \Delta \mu \cdot R \gtrsim 1 \), when the calculation of \( H_{\text{int}} \) is dominated by a vicinity of one soliton only.

5. Conclusion

In this work, we have derived analytical expressions for an effective potential of interaction between two- and three-dimensional solitons (including the case of the two-dimensional vortex solitons) belonging to two different modes which are incoherently coupled through cross-phase modulation in models of media with the self-focusing cubic and self-defocusing quintic nonlinearities. The derivation was based on the calculation of the interaction term in the full Hamiltonian of the system. An essential peculiarity is that, in the 3D case, as well as in the case of 2D solitons with unequal masses, the main contribution to the interaction potential originates from a vicinity of one or both solitons, similarly to what was recently found in the 2D and 3D single-mode systems [4], while in the case of identical 2D solitons, the dominating area covers all the space between the solitons. Except for the case of the interaction between light and heavy solitons, which may have either sign, the solitons always attract each other.
The attraction between the solitons may give rise to their orbiting bound states in the 2D and 3D cases (in the latter case, it is assumed that the two solitons move in one plane). Orbiting of incoherently interacting 2D solitons was experimentally observed in a photorefractive medium [1]. Numerical simulations and analytical arguments presented in Refs. [3,4] demonstrate that the orbiting bound states of the 2D solitons in the single-mode systems, including the SHG model, are unstable. However, it was recently pointed out in Ref. [10] that they might be stable in the bimodal system. Indeed, for the orbiting state, the interaction potential (11), (13), or (15) must be supplemented by the centrifugal energy $E_{cf} = (M^2/2m)R^{-2}$, where $M$ is the angular momentum of the soliton pair, and $m$ is the soliton’s effective mass. A straightforward consideration of the net potential, $H_{int} + E_{cf}$, demonstrates that it may have two stationary points, the one corresponding to a smaller value of $R$ being a potential minimum that gives rise to a stable orbiting state. The instability of similar states in the single-mode systems is due to the fact that, in those systems, the interaction potential also depends on the phase difference between the solitons, the effective mass corresponding to the phase-difference degree of freedom being negative [14]. This, eventually, made the existence of stable stationary points of the effective interaction potential impossible.

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