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Nonlinear magnetoinductive waves and domain walls in composite metamaterials

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Abstract

We describe novel physics of nonlinear magnetoinductive waves in left-handed composite metamaterials. We derive the coupled equations for describing the propagation of magnetoinductive waves, and show that in the nonlinear regime the magnetic response of a metamaterial may become bistable. We analyze modulational instability of different nonlinear states, and also demonstrate that nonlinear metamaterials may support the propagation of domain walls (kinks) connecting the regions with the positive and negative magnetization.

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The past decade observed many advances in the design and engineering of artificial structures with unique electromagnetic response, which have broadened significantly the range of possible wave phenomena that can be accessed in experiment. In particular, it has been shown that the composite structures may allow realizing the materials with simultaneously negative dielectric permittivity and magnetic permeability, also known as left-handed media [1], the unique materials because of their surprising and often counterintuitive electromagnetic properties. The composite metallic structures consisting of arrays of wires and split-ring-resonators (SRRs) have been demonstrated to possess left-handed properties in the microwave frequency range [2].

A standard theoretical approach for analyzing the properties of composite metamaterials is based on the effective medium approximation for both linear [3–5] and nonlinear [6,7] metamaterials. In such approach a microstructured composite is treated as a homogeneous isotropic medium characterized by effective macroscopic parameters. This approximation is justified when the characteristic scale of the wavelength of the electromagnetic field is much larger than the period of the microstructured medium. Moreover, this effective medium approach is based on a simple averaging over the lattice of micro-elements, and usually it does not take into account any internal (or eigen) modes of the structure, which might exist due to interaction between the individual elements via microscopic electromagnetic fields. Such eigenmodes in the material containing arrays of SRRs are referred to as magnetoinductive waves [8].

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In this paper we study a novel physics of nonlinear composite metamaterials stipulated by magnetic interaction between SRRs. We consider a cubic lattice of parallel SRRs and demonstrate that the effective coupling between the resonators is highly anisotropic. We derive the discrete coupled equations and describe the properties of linear and nonlinear magnetoinductive waves. We show that in the nonlinear regime the magnetic response of a metamaterial may become bistable, and we analyze modulational instability of different nonlinear states. We also demonstrate that nonlinear metamaterials may support the propagation of domain walls (kinks) connecting the regions of positive and negative magnetization, and simulate their dynamics numerically.

1. Model

We consider a three-dimensional cubic lattice of identical parallel SRRs, as shown in Fig. 1(a). We assume that the SRR slits are infilled with a nonlinear dielectric with a Kerr-like nonlinear response, and with dielectric permittivity $\varepsilon = \varepsilon_l + \alpha|E|^2/|E_c|^2$, where E_c is the characteristic nonlinear field, and $\alpha = \pm 1$ correspond to the focusing and defocusing nonlinearity, respectively; the case $\alpha = 0$ describes a linear response.

An external electromagnetic field induces currents in SRRs; this generates additional magnetic fields which determine an overall magnetic response of the composite structure. In this paper, we take into account the interaction between the nearest neighboring magnetic resonators induced by microscopic fields, as shown in Fig. 1(b). This approximation describes the basic physics due to the SRR interaction, since the magnetic field of each SRR decays fast enough and the interaction between the next neighboring magnetic resonators is weak. In linear regime, it is possible to take into account interaction between all resonators in the metamaterial [8], however in the nonlinear problem the nearest neighbour approximation simplifies the solution.

We use the indices n, q, m to denote the position of the resonators along the axis x, y, z, respectively. Each SRR represents an effective oscillatory circuit with inductance $L = 4\pi a[\ln(8a/r) - 7/4]/c^2$ of the loop, nonlinear capacitance $C_{NL} \approx \varepsilon(|E_g|^2)r^2/4d_g$ of the slit, and resistance $R = 2\pi a/\sigma S_{eff}$ of the SRR wire (see, e.g. Refs. [7,9] and references therein), where a is the radius of SRR, r is the radius of the wire, d_g is the size of the SRR slit, E_g is the electric field induced in the SRR slit, σ is the conductivity of the SRR wire, S_{eff} is the effective cross-section of the wire, and c is the speed of light.

In the system of magnetically interacting SRRs, we also have an additional mutual inductance, described by the matrix $\hat{M}$, acting between the neighboring resonators. The external electromotive force $\mathcal{E}_{n,q,m}$ in each oscillatory circuit is determined by the external magnetic field H_0, $\mathcal{E}_{n,q,m} = -(\pi a^2/c)(dH_0/dt)_{n,q,m}$. Electromotive force induced in the resonator at the site (n, q, m) due to the mutual inductance with the other resonators can be written as follows:

$$e_{n,q,m} = \sum_{n',q',m'} M_{n'-n,q'-q,m'-m} \frac{dI_{n',q',m'}}{dt}, \tag{1}$$

where $M_{n'-n,q'-q,m'-m}$ is the elements of the mutual inductance matrix describing the interaction between the resonators at the sites (n, q, m) and (n', q', m');

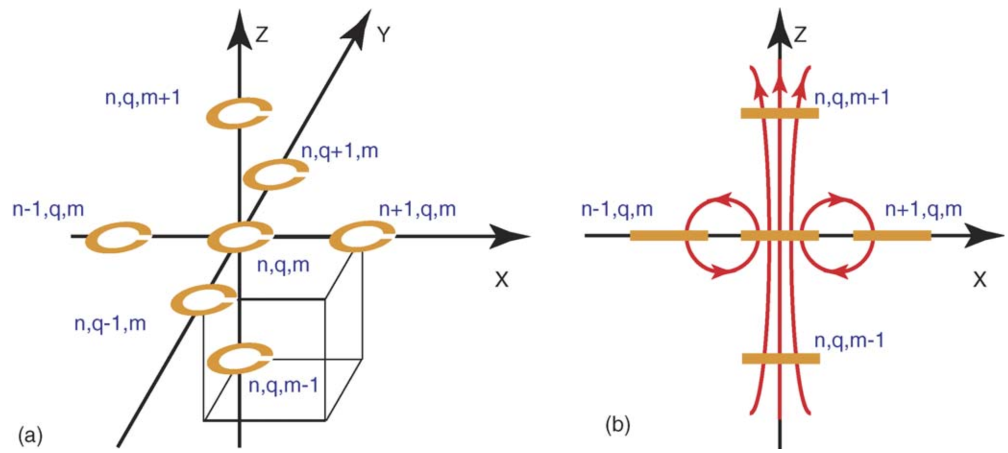

Fig. 1. (a) Three-dimensional geometry and (b) side view of a cubic lattice of split-ring-resonators. Lines in (b) show the direction of microscopic magnetic field.

$M_{0,0,0} = 0$. Using symmetry, we can present all nonzero elements of the matrix M as follows:

$$M_{1,0,0} = M_{-1,0,0} = M_{0,1,0} = M_{0,-1,0} = M_{\perp}, \tag{2}$$

$$M_{0,0,1} = M_{0,0,-1} = -M_{||}, \tag{3}$$

and derive a set of coupled equations governing the dynamics of the electric currents in SRRs:

$$L\frac{dI_{n,q,m}}{dt} + RI_{n,q,m} + U_{n,q,m} = \mathcal{E}_{n,q,m} + M_{\perp}\frac{d}{dt}\left[I_{n+1,q,m} + I_{n-1,q,m} + I_{n,q+1,m} + I_{n,q-1,m}\right] - M_{||}\frac{d}{dt}\left[I_{n,q,m+1} + I_{n,q,m-1}\right], \tag{4}$$

$$C_{NL}(|U_{n,q,m}|^2)\frac{dU_{n,q,m}}{dt} = I_{n,q,m}, \tag{5}$$

where $U_{n,q,m}$ is the voltage across a slit of the corresponding resonator. Capacitance of the SRR slit is a sum of linear and nonlinear parts, $C_{NL}(|U_{n,q,m}|^2) = C_0 + \Delta C_{NL}(|U_{n,q,m}|^2)$, and we assume that the nonlinear part is much smaller than the linear one.

We assume that the fields vary harmonically in time [i.e. $\sim\exp(i\omega t)$] neglecting the generation of higher harmonics, and use the approximation of the slowly-varying amplitudes for analyzing Eqs. (4) and (5). The magnetic momentum of each SRR is proportional to the current in this resonator, $\mathbf{m}_{n,q,m} = \mathbf{z}_0\pi a^2\mathcal{I}_{n,q,m}/2c$, where $\mathbf{z}_0$ is the unit vector along the z-axis, and $\mathcal{I}_{n,q,m}$ is the amplitude of the harmonic current $I_{n,q,m}$. We are interested mostly in the magnetic response of the structure near the SRR resonant frequency, $\omega_0 = 1/\sqrt{LC_0}$. Our choice is motivated by two reasons: (i) nonlinear effects are essentially enhanced for the frequencies close to the resonant frequency, and (ii) the left-handed properties of the composite of wires and SRRs are observed for the frequencies on the right-hand side from the SRR resonance. Eq. (4) gives an explicit relation between the amplitudes of the current in SRR, $\mathcal{I}_{n,q,m}$, and the voltage applied across the SRR slit, $\mathcal{U}_{n,q,m}$:

$$\mathcal{U}_{n,q,m} = \mathcal{I}_{n,q,m}/i\omega[C_0 + \Delta C_{NL}(|U_g|^2)]. \tag{6}$$

Next, we introduce the dimensionless variables $\tau = \omega_0 t$, $\Omega = (\omega - \omega_0)/\omega_0$, and $\Psi_{n,q,m} = \mathcal{I}_{n,q,m}/\mathcal{I}_c$, where $\mathcal{I}_c = \omega_0 C_0 U_c$ is the characteristic nonlinear current, $U_c = E_c \times d_g$ is the characteristic voltage, $\kappa_{||,\perp} = M_{||,\perp}/L$ are the coupling coefficients, $\gamma = R/L\omega_0$ is the damping coefficient, and $\Sigma_{n,q,m} = -\omega H_0\pi a^2/c\omega_0 L\mathcal{I}_c$ is the normalized electromotive force. The coupling coefficients $\kappa_{||,\perp}$ can be calculated numerically for any SRR geometry, and the magnetic field of the loop current is well known (see, e.g. Ref. [10]). However, here we use an approximate expression for the magnetic field of the loop current [10], which yields $\kappa_{\perp} \approx (1/2)(a/d)^3 = \kappa$ and $\kappa_{||} \approx 2\kappa$. In the dimensionless units, the equations for the slowly varying amplitude of the current in SRR (and, respectively, the magnetic momenta) can be written as:

$$i\frac{d\Psi_{n,q,m}}{d\tau} - \{2\Omega - i\gamma + \alpha|\Psi_{n,q,m}|^2\}\Psi_{n,q,m} - \Sigma_{n,q,m} = 2\kappa\{\Psi_{n,q,m+1} + \Psi_{n,q,m-1} - 2\Psi_{n,q,m}\} - \kappa\{\Psi_{n+1,q,m} + \Psi_{n-1,q,m} + \Psi_{n,q+1,m} + \Psi_{n,q-1,m} - 4\Psi_{n,q,m}\}, \tag{7}$$

where $\tau = \omega_0 t$.

In addition, Eq. (7) may include long-range effects due to the interaction with next-neighboring and other SRRs; such effects will produce additional terms in the right-hand-side of Eq. (7) in the form of higher-order discrete derivatives, and they can be shown to remain small not changing the major conclusions of our analysis.

Eq. (7) have a clear physical meaning. The coupling coefficient κ determines a shift of the oscillator eigenfrequency due to an effective coupling between SRRs, the nonlinear term produces an eigenfrequency shift due to the nonlinear self-action effect. The effect of the higher-order derivatives will be an additional shift of the resonant frequency. The right-hand side of Eq. (7) represents the second-order difference operators which determines the character of the wave propagation and diffraction. The opposite signs between the respective discrete difference operators make the derived equations fundamentally different from those describing the dynamics of two-dimensional discrete systems where the difference operators have the same sign.

2. Linear magnetoinductive waves

First, we describe linear magnetoinductive waves [8] in lossless metamaterial, i.e. we assume $d/d\tau = 0$, $\alpha = 0$, $\gamma = 0$, and $\Sigma_{n,q,m} = 0$, and substitute $\Psi_{n,q,m} = F(\Omega, \mathbf{k})\exp(-ink_x d - iqk_y d - imk_z d)$, where

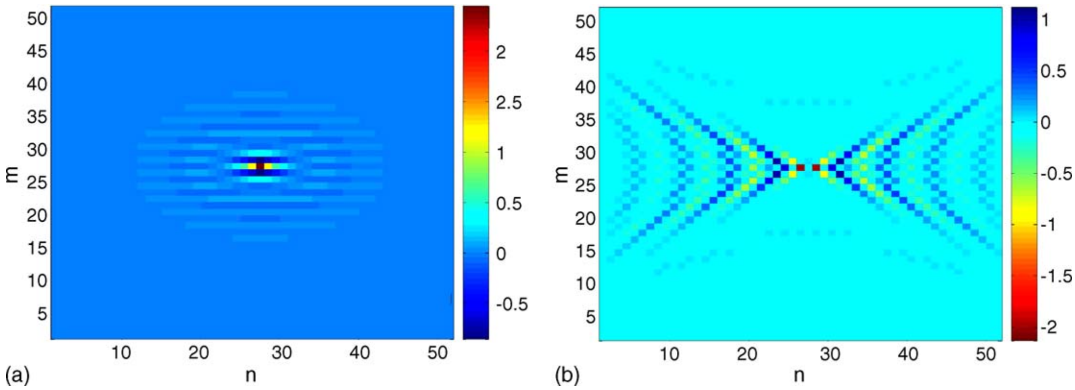

Fig. 2. Two-dimensional anisotropic discrete diffraction of linear magnetoinductive waves in a lattice of SRRs. Shown is the SRR magnetization on the plane (n, m) (a) at the upper cutoff frequency when $\Omega = 3\kappa$, and (b) at the resonant frequency when $\Omega = 0$.

$k_{x,y,z}$ are the Cartesian components of the wavevector **k**. The corresponding dispersion relation takes the form:

$$\Omega = 4\kappa \sin^2\left(\frac{k_z d}{2}\right) - 2\kappa\left[\sin^2\left(\frac{k_x d}{2}\right) + \sin^2\left(\frac{k_y d}{2}\right)\right],$$

and its real solutions describe the linear waves with the frequencies $|\Omega| < 4\kappa$. For the dimensional frequencies, this relatively narrow region is centered at the SRR eigenfrequency. We note that the upper cutoff of the frequency band of linear waves corresponds to the edge of the Brillouin zone at $k_x = k_y = 0$ and $k_z = \pi/d$, and in this limit the linear waves are longitudinal (magnetic plasmons). The lower cutoff of the frequency band corresponds to the point $k_z = 0$, $k_x = k_y = \pi/d$, and the corresponding magnetoinductive linear waves are transverse. At the resonance ($\Omega = 0$) and in the vicinity of the origin of the Brillouin zone, the isofrequency surface forms a resonant cone defined as $2(k_z d)^2 = (k_x d)^2 + (k_y d)^2$, and the phase velocity of the linear magnetization waves becomes orthogonal to their group velocity.

To study the generation and propagation of the magnetoinductive waves in the SRR lattices, we perform numerical simulations of Eq. (7). For simplicity, we consider a two-dimensional geometry when all fields are homogeneous along the y-axis. We simulate a lattice of 51×51 array of SRRs exciting only one SRR in the middle of the structure (at $n = m = 26$), taking the damping coefficient $\gamma = 0.01$ and different excitation frequencies, $\Omega = 0$ and $\Omega = 3\kappa$. To suppress the wave reflection from the boundaries of the numerical domain, we increase smoothly the damping coefficient in the three rows of resonators near the boundaries. At the upper cutoff frequency, $\Omega = 3\kappa$, the longitudinal wave is shown in Fig. 2(a). In the case of the resonance, $\Omega = 0$, we observe a resonant cone as shown in Fig. 2(b).

3. Nonlinear magnetization

To study nonlinear effects, we consider a metamaterial illuminated by a homogeneous electromagnetic wave. The external electromotive force is the same in all SRRs: $\Sigma_{n,q,m} = \Sigma_0$. Steady homogeneous magnetization of the SRR lattice ($\Psi_{n,q,m} = \Psi_0$) can be obtained from Eq. (7) and it is described by the nonlinear dispersion, which relates magnetization amplitude with the external magnetic field:

$$\{(2\Omega + \alpha|\Psi_0|^2)^2 + \gamma^2\}|\Psi_0|^2 = |\Sigma_0|^2. \quad (8)$$

Two examples of the dependence of Ψ_0 on Σ_0 are shown in Fig. 3. When $\Omega\alpha < 0$, this dependence becomes multi-valued [see Fig. 3(a)] and, as is shown below, the middle branch corresponds to unstable solution. Fig. 3(b) shows the dependence of Ψ_0 on Σ_0, when $\Omega\alpha > 0$.

Such a bistable dependence of the metamaterial parameters may allow switching between the left-handed transparent and right-handed opaque states by applying a varying external magnetic field [7], and may support the existence of dynamically induced transparent regions inside an opaque material slab and the formation of spatiotemporal electromagnetic solitons.

4. Modulational instability

Next, we study linear stability of the homogeneous nonlinear states (8) with respect to small perturbations of the form, $\delta\Psi_{n,q,m} = (u + v)e^{\lambda t} + (u^* - v^*)e^{\lambda^* t}$, where $u = A_1 \exp(ik_x nd + ik_y qd + ik_z md)$ and $v =$

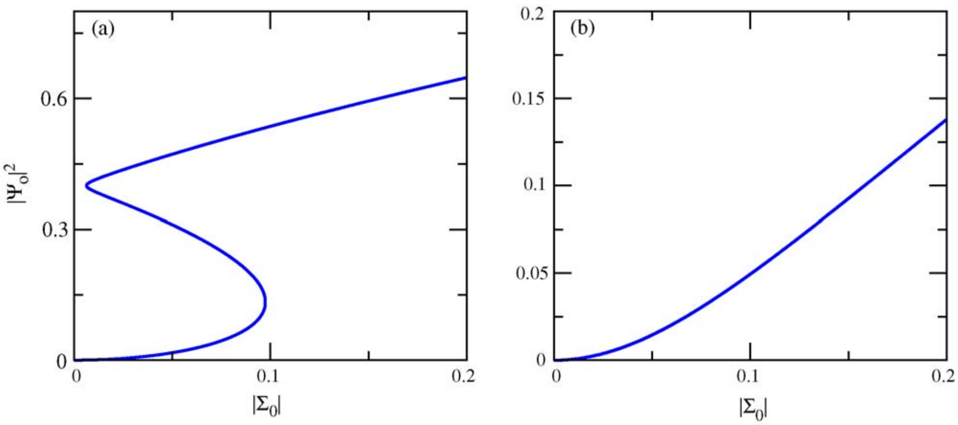

Fig. 3. Dependence of the absolute value of homogeneous magnetization $|\Psi_0|$ of the metamaterial on the absolute value of the external field $|\Sigma_0|$ for two cases: (a) $\Omega = 0.2$, $\alpha = -1$, (b) $\Omega = 0.2$, $\alpha = +1$.

$A_2 \exp(-ik_x nd - ik_y qd - ik_z md)$. Substituting $\Psi_{n,q,m} = \Psi_0 + \delta\Psi_{n,q,m}$ into Eq. (7), we obtain the following expression for the instability growth rate λ:

$$\lambda = -\gamma + \sqrt{-3|\Psi_0|^4 - 8\alpha\bar{\Omega}|\Psi_0|^2 - 4\bar{\Omega}^2}, \quad (9)$$

where $\bar{\Omega} = \Omega - 2\kappa[2\sin^2(k_z d/2) - \sin^2(k_x d/2) - \sin^2(k_y d/2)]$ can take any value in the range $[\Omega - 4\kappa, \Omega + 4\kappa]$. A simple analysis show that the real part of λ can become positive and, therefore, the modulational instability can occur when $\alpha\bar{\Omega} < 0$. In this case, the boundaries of the modulational instability region are defined from the relations:

$$|\Psi_0|^2 = \left(\frac{4\bar{\Omega}}{3}\right) \pm \sqrt{\left(\frac{4\bar{\Omega}^2}{9}\right) - \left(\frac{\gamma^2}{3}\right)}. \quad (10)$$

Both the instability and bistability regions can be presented on the parameter plane (Ψ_0, Ω), as shown in Fig. 4 for the parameters $\alpha = -1$ and $\kappa = 0.0025$. The case $\alpha = +1$ can be recovered from the same results by changing $\Omega \to -\Omega$. The curve that depicts the instability region consists of three parts. The upper part represents the boundary (10) for $\bar{\Omega} = \Omega + 4\kappa$, which corresponds to the instability with respect to the excitation of the transverse waves (at $k_z = 0$, $k_x = k_y = \pi/d$). The lower curve corresponds to the excitation of the longitudinal waves (at $k_z = \pi/d$, $k_x = k_y = 0$). Along the horizontal line the instability occurs for the waves propagating at some angle to the main axis of the lattice. Thus, the instability covers completely the region corresponding to the middle branch of the bistable dependence shown in Fig. 3(a). Some parts of the upper and lower branches of that dependence are also unstable. The instability also occurs for $\alpha\Omega > 0$, as shown in Fig. 4 (shaded region).

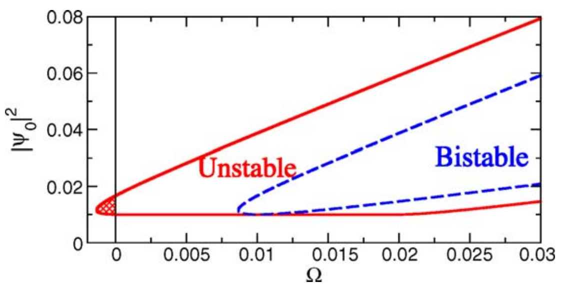

Fig. 4. Instability region (bounded by a solid curve) on the parameter plane (Ψ_0, Ω). Dashed: the region of the decreasing branch of the homogeneous magnetization shown in Fig. 3(a). Shaded: instability domain for the magnetization with non-bistable behavior for $\alpha\Omega > 0$ [see Fig. 3(b)].

Existence of unstable regions can be crucial for engineering tunable metamaterials with nonlinear properties. The magnetization waves propagating inside the metamaterial represent alternating positively and negatively magnetized SRRs. As a result, the average magnetization of the metamaterial will be dramatically reduced, possibly suppressing the regions of the negative effective permeability.

5. Domain walls

Finally, we study numerically nonlinear switching waves (or domain walls) which can propagate in the nonlinear metamaterial. We solve Eq. (7) numerically for a two-dimensional SRR lattice with the size $N \times N$, when $N = 100$, for defocusing nonlinearity ($\alpha = -1$) and $\Omega = 0.2$, when no linear waves exist. First, we create an inhomogeneous magnetization by a step-like external field $\Sigma_n = \Sigma_0 + \delta\Sigma \times \chi(n - N/2)$,

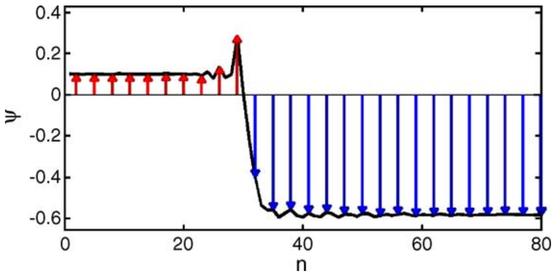

Fig. 5. Example of a switching wave of magnetization (kink) in a nonlinear metamaterial. Shown is the magnetization $\Psi = \Psi_{n,0,0}$. Arrows show schematically the magnetization of the corresponding parts of the metamaterial, i.e. the magnetization is positive for arrows directed upwards, and negative, otherwise.

where $\chi = -1$ for $n < N/2$, and $\chi = +1$, otherwise. We chose Σ_0 and $\delta\Sigma$ in such a way that the lower state excites a homogeneous magnetization of the metamaterial corresponding to the lower branch of Fig. 3, while the higher state excites the upper branch. After a steady state is reached, we switch off the inhomogeneous part (put $\delta\Sigma = 0$), and observe the propagation of a nonlinear switching wave (kink) with the profile shown in Fig. 5. Choosing the parameters Σ_0 and $\delta\Sigma$, we may control the speed and direction of the kink propagation. Two out-of-phase kinks may create a magnetization domain which differs from the rest of the material. Depending on the external field, such a domain can collapse, expand, or preserve its shape. The possibility to control creation and dynamics of such domains seems to be promising for the design of the structures with controllable periodic magnetization, photonic crystals, which parameters can be made tunable.

In conclusion, we have analyzed, for the first time to our knowledge, nonlinear magnetoinductive waves in composite left-handed metamaterials. We have derived the effective discrete model that describes the propagation of magnetoinductive waves in a lattice of split-ring-resonators, and studied both linear and nonlinear waves demonstrating that they can change dramatically the average magnetization of left-handed materials. We have revealed that the bistable magnetic response may lead to the propagation of magnetization domain walls in nonlinear metamaterials.

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