I. INTRODUCTION

Periodic photonic structures offer a rich platform to investigate the propagation properties of light beams, beam steering, and manipulation (for a comprehensive review, see Refs. [1,2]). Discrete waveguide arrays can support a special type of nonlinear localized waves, namely, discrete or lattice solitons. Another type of interesting solitons, namely, multipole-mode lattice solitons, have been investigated both in theory [3,4] and experiments [5,6]. Recently, it was shown theoretically [7–9] and demonstrated experimentally [10–12] that nonlinearity-induced self-trapping of light may become possible near the edge of a one-dimensional semi-infinite waveguide array, leading to the formation of discrete surface solitons (see also the review papers [13,14]). Soon afterward, the study on surface solitons has been extended to two-dimensional (2D) photonic structures. Thus, it is shown that 2D surface solitons could exist at the edges and corners of finite 2D photonic lattices [15], which were demonstrated by two remarkable experiments in optically induced lattices in photorefractive crystal [16] and fs-laser-written arrays in bulk-fused silica [17]. Interestingly, it is shown that surface solitons can be set in rotation along a certain ring in two-dimensional circular guiding structures [18,19]. Note that surface solitons can be formed at the interface between two dissimilar periodic media [15,20–22].

Surface solitons have been normally considered for one-frequency modes propagating in cubic or saturable nonlinear media; however, it is reported that photonic lattices in quadratic nonlinear media can support lattice solitons, the so-called multicolor lattice or discrete solitons [23,24]. It should be pointed out that so far, surface discrete or lattice solitons have been demonstrated experimentally in periodically poled lithium niobate waveguide arrays but only in semi-infinite waveguide arrays [25]. More elaborated theory of one-dimensional surface solitons in truncated quadratic nonlinear photonic lattices, the so-called two-color surface lattice solitons, has been developed recently in Ref. [26], which showed that new classes of one-dimensional two-color twisted surface solitons are stable in a large domain of their existence. Very recently, it is shown that parametric localization of light at an interface separating two different one-dimensional quadratic nonlinear lattices is possible, and the strategy on control and manipulation of interface solitons by tuning the lattice and waveguide parameters has been discussed [27]. However, two-color surface solitons in 2D quadratic nonlinear lattice are rarely studied [28]. Two-dimensional geometries may support even more rich and interesting soliton entities with engineering phases, which will be the main aim in this work.

We thus study the propagation of nonlinear multipole modes along an interface, separating two distinct optical lattices imprinted in two-dimensional nonlinear quadratic media. We analyze the impact of guiding parameters of lattices on the existence and stability of multipole-mode interface solitons in different phase mismatching conditions. Remarkably, our results show that multipole-mode interface solitons have the highly asymmetric shape and are stable in the broad range of system parameters.

The paper is organized as follows. In Sec. II, we introduce the system governing the parametric interaction of light beams in photonic lattices and numerical methods applied for our study. Section III is devoted to the analysis of multipole-mode interface solitons in nonlinear quadratic nonlinear lattices, in particular, we focus on the properties of dipole- and quadrupole-mode interface solitons. Section IV concludes the paper.

II. MODEL

We consider the propagation of light at the interface between two dissimilar optical lattices imprinted in quadratic nonlinear media, which involves the interaction between fundamental frequency (FF) and second-harmonic (SH) waves. Light propagation is described by the following coupled nonlinear equations [24]:

\[
\begin{align*}
\frac{\partial q_1}{\partial z} &= \frac{d_1}{2} \left( \frac{\partial^2 q_1}{\partial x^2} + \frac{\partial^2 q_1}{\partial y^2} \right) - q_1^* q_2 \exp(-i\beta z) - pR(x,y)q_1, \\
\frac{\partial q_2}{\partial z} &= \frac{d_2}{2} \left( \frac{\partial^2 q_2}{\partial x^2} + \frac{\partial^2 q_2}{\partial y^2} \right) - q_1^2 \exp(i\beta z) - 2pR(x,y)q_2,
\end{align*}
\]

where \(q_1\) and \(q_2\) represent the normalized complex amplitudes of the FF and SH fields, \(x\), \(y\), and \(z\) stand for the...
normalized transverse and longitudinal coordinates, respectively, \( \beta \) is the phase mismatch, and \( d_1 = -1, \ d_2 = -0.5; \ p \) is the lattice depth; the function \( R(x,y) = H(x) \sin^2(Kx) \sin^2(Ky) \) describes the profile of waveguide formed with two dissimilar lattices with space modulation frequency \( K \), in which \( H(x) = 1 \) at \( x \geq 0 \), and \( H(x) = s \) at \( x < 0 \), where \( s \) is the relative lattice depth for the left-side one. A typical profile for such waveguide is shown in Fig. 1, in which one can see clearly that an interface is created by two distinct lattices [Fig. 1(b)], in comparison with uniform periodic waveguide arrays [Fig. 1(b)]. The system of Eq. (1) conserves the total power

\[
U = U_1 + U_2 = \int_{-\infty}^{\infty} (|q_1|^2 + |q_2|^2) \, dx \, dy. \tag{2}
\]

The stationary solutions for the lattice-supported surface solitons can be found in the form \( q_1(x,y,z) = u(x,y) \exp(i b_1 z) \), and \( q_2(x,y,z) = v(x,y) \exp(i b_2 z) \), where \( u(x,y) \) and \( v(x,y) \) are real functions, and \( b_{1,2} \) are real propagation constants satisfying \( b_2 = \beta + 2b_1 \). Substitution of the above expressions into Eq. (1) yields the following system of equations:

\[
\frac{d_1}{2} \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) - uv + b_1 u - pR(x,y)u = 0,
\]

\[
\frac{d_2}{2} \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) - 2uv + b_2 v - 2pR(x,y)v = 0. \tag{3}
\]

We solved the system of coupled equation (3) numerically by using the standard relaxation method. Families of interface solitons are determined by the propagation constant \( b_1 \), the lattice depth \( p \) and \( s \), and the phase mismatch \( \beta \). For simplicity, we set the modulation \( K = 2 \) and vary other parameters. To investigate the stability of two-color interface solitons, we have applied the beam propagation method to perform the extensive numerical simulations of the evolution governed by Eq. (1) with the input conditions \( q_1(z=0) = u(x,y)[1 + p_1(x,y)] \), and \( q_2(z=0) = v(x,y)[1 + p_2(x,y)] \), where \( u(x,y) \) and \( v(x,y) \) are the stationary solutions of Eq. (1), and \( p_{1,2} \) are random functions with Gaussian distribution and variance \( \sigma_{\text{noise}}^2 = 0.01 \). We have propagated the perturbed multipole-mode interface solitons over hundreds of units for a broad range of system parameters.

### III. MULTIPOLAR-MODE INTERFACE SOLITONS

#### A. Dipole-mode solitons

The waveguide arrays shown in Fig. 1 support rich families of two-color lattice solitons. Here, we are interested in multipole-mode solitons with certain engineering phases formed along the interface between two distinct lattices. The simplest example of two-color interface multipole modes is the interface dipole-mode soliton, which is composed of two bright spots with a \( \pi \)-phase difference for the FF field and in phase for SH field, respectively. Note that the phase jump is located in two sides of the photonic lattices [Fig. 2, where the dashed vertical lines indicate the interface position]. As shown in Figs. 2(c) and 2(d), one notes that the profiles of such interface modes are highly asymmetric due to the non-uniform of two distinct lattices [Fig. 1(b)]. For comparison, Figs. 2(a) and 2(b) show the symmetric dipole-mode lattice solitons supported in a homogeneous periodic lattice with \( s = 1 \) [Fig. 1(a)]. It is noted in Fig. 2 that the profiles of SH field are more localized than those of FF field.

As shown in Fig. 3(a), the total power of dipole-mode interface solitons is a nonmonotonic function of the propagation constant. There exists a cutoff of propagation constant \( (b_{\text{co}}) \). The total power \( U \) diverges as the propagation constant \( b_1 \) approaches the cutoff value. It is shown that the formation of dipole-mode interface solitons requires higher power than their counterpart in homogeneous lattices [compare the curve 3 with \( s = 1 \) with curve 1 with \( s = 0.6 \) in Fig. 3(a)]; in other words, the asymmetric modes require higher powers for their formation than the symmetric ones. Also note that the phase mismatch, resulting from the parametric interaction between two fields, alters the minimum power for the formation of interface solitons [compare curves 1 and 2 in Fig. 3(a)]. It is interesting to note that the powers of dipole-mode interface
The multipole-mode interface solitons are identical in curves 2 and 3 in Fig. 3(a), but it should be noted that they present different types of solitons, i.e., curve 2 presents the asymmetric modes with $s=0.6$ and $\beta=3$, while curve 3 is for the symmetric modes with $s=1$ and phase matching $\beta=0$. The existence domain of two-color dipole-mode interface solitons is determined by the waveguide parameters; thus, the required minimum power for the formation varies depending on the lattice period and the phase mismatching conditions [Fig. 3(c)].

As mentioned above, the dipole-mode interface solitons have highly asymmetric profiles, as two distinct lattices forming the interface require different peak powers for the self-sustained action of light beams. The power distribution to measure the degree of asymmetry for the profiles in dipole-mode interface solitons is defined by the ratio $E_{1}=U_{1,1}/U_{1,1}$ for FF beam and $E_{2}=U_{2,2}/U_{2,2}$ for SH field, where $U_{1,1}$ and $U_{2,2}$ are the powers located at $x<0$, and where $U_{1,1}$ are the powers located at $x>0$. As shown in Fig. 3(b), the ratio $E_{1,2}$ changes dramatically as the propagation constant approaches to its cutoff, which means that the profiles of dipole-mode interface solitons with lower powers become more asymmetric for a given lattice depth $s$. For a certain total power, it is shown that the shape of dipole-mode interface solitons varies dramatically, depending on the guiding parameter $s$ [Fig. 3(d)]. It is also interesting to note that the degree of asymmetry in the profiles of interface solitons is less influenced for SH field than FF field.

Our results show that there exists a narrow instability region near the cutoff of propagation constant, as shown with dotted curves in Fig. 3(c). However, above a certain power, the dipole-mode interface solitons are completely stable. We have found that the width of narrow instability domain almost remains unchanged with the change in guiding parameter $s$, which means that the asymmetry of the soliton profiles does not substantially change the instability scenarios. Thus, Figs. 4(a)–4(c) show a decay of unstable dipole-mode interface soliton, which typically emits the most of power in the right side of lattices, and eventually transforms into a fundamental surface lattice soliton, which is the most robust entity in the system. One of the important results in this paper is that highly asymmetric dipole-mode interface solitons become completely stable when their power is above a certain value [Fig. 3(a)]. In Figs. 4(d)–4(f), we show a stable propagation dynamics of highly asymmetric dipole-mode soliton propagating along the interface in quadratic nonlinear photonic lattices.

### B. Quadrupole-mode solitons

Besides the dipole-mode interface solitons discussed above, we find that the quadratic nonlinear waveguide also
supports more complex multipole-mode interface solitons. Figure 5(a) shows an example of quadrupole-mode solitons, where FF field features the out-of-phase combination among four bright spots, two of which are located in the opposite side of the lattice, while SH field features the in-phase combination of four bright spots. Similar to the properties of dipole-mode interface solitons, the total power of quadrupole-mode interface solitons is also a nonmonotonic function of the propagation constant, and there exists the cutoff value \( b_{co} \) of the propagation constant, below which the interface solitons are terminated. It is shown that the required minimum power for the formation of quadrupole-mode interface solitons depends on the guiding parameter \( s \) [Fig. 6(c)], i.e., the higher power is required for the formation of more asymmetric solitons. Note that the propagation constant cutoff \( b_{co} \) is depending on the phase mismatch [Fig. 6(d)]. As shown in Fig. 6(b), the ratio \( E_{1,2} \) changes dramatically as the propagation constant approaches to its cutoff, which means that the profiles of solitons become more asymmetric for a given lattice depth \( s \).

We have also found that highly asymmetric quadrupole-mode interface solitons become completely stable in quadratic nonlinear lattices when the propagation constant is above a certain value [Fig. 6(a)], below which interface solitons are unstable in a very narrow region near the cutoff of propagation constant [as shown with dotted curves in Fig. 6(a)]. Thus, Figs. 7(a)–7(c) show a decay of unstable quadrupole-mode interface soliton, which emits the most of...
its power, and finally transforms into a fundamental interface lattice soliton, which is the most robust state in the system. However, it is interesting to note that highly asymmetric quadrupole-mode soliton can stably propagate along the interface in quadratic nonlinear photonic lattices over a long distance, which exceeds any experimentally accessible distance.

IV. CONCLUSIONS

We have shown that the multipole-mode solitons can be created at the interface formed between two distinct lattices in quadratic nonlinear media. We have analyzed the impact of the guiding parameters of the lattice and the phase mismatch on the properties and stability of interface multipole modes. Our results reveal that highly asymmetric multipole-mode interface solitons are stable in a broad region of system parameters. It is shown that the asymmetry of the interface solitons can be controlled by tuning the guiding parameters of the lattices; thus, it may open a possible way to control the flow of light in tunable photonic lattices. The concept reported here could be extended to other systems such as Bose-Einstein condensate and atomic optics.

ACKNOWLEDGMENTS

The author acknowledges useful discussions with Professor Yuri Kivshar. This work was supported by the Australian Research Council.


