Comment on “Nonexistence of Small-Amplitude Breather Solutions in $\phi^4$ Theory”

In a recent Letter,\(^1\) a proof of nonexistence of small-amplitude breather solutions to the nonlinear equation

$$u_{tt} - u_{xx} + u - \frac{1}{6} u^3 = 0$$

was given. This equation has a formal breatherlike (localized oscillating) solution in the form of an asymptotic power series in $S = \epsilon \text{sech}(\epsilon x)$, where $\epsilon = (2 - \omega^2)^{1/2} \ll 1$, $\omega$ being the breather’s internal frequency; however, the series does not converge to a genuine solution, and, in fact, the “breather” very slowly fades because of emission of radiation. A corresponding energy emission rate has been demonstrated to be exponentially small in $\epsilon$. Similar results were obtained earlier by Eleonsky et al.\(^2\)

The method of Ref. 1, based on matched asymptotic expansions, is fairly general and may be applied to other equations with a polynomial nonlinearity. In this Comment we aim to draw attention to an equation which has important physical applications,

$$u_{tt} - u_{xx} + u - \frac{1}{6} u^3 = 0.$$  \hspace{1cm} (2)

In the lowest approximation, breatherlike solutions to (2) are

$$u \approx 4 \epsilon \sin[(1 - \epsilon^2/2)t] \text{sech}(\epsilon x).$$  \hspace{1cm} (3)

Equation (2) does not have exact breather solutions either, and the rate of energy emission from the approximate breather (3) can be found by the method of Segur and Kruskal\(^1\) (see also Ref. 2). However, it is interesting to note that the result can be obtained in another way with the aid of the perturbation theory for the sine-Gordon equation. Indeed, Eq. (2) may be rewritten in the form

$$u_{tt} - u_{xx} + u = \left(1/5\right)u^5 - \left(1/7!\right)u^7 + \cdots.$$  \hspace{1cm} (4)

The rate of energy emission from the sine-Gordon small-amplitude breather (3) under the action of the perturbation $au^3$ ($a \ll 1$) has been calculated by one of us in a recent paper:\(^3\)

$$W = C(\sqrt{2}/5)(64\pi/3)^2 \exp(-2\sqrt{2}\pi/\epsilon),$$  \hspace{1cm} (5)

with the radiation frequency $\omega \approx 3$; $C$ is given by an infinite sum $1 + C_1 + C_2 + \cdots$, where the constants $C_j$, though produced by higher terms of expansion in powers of $\epsilon$, are formally all of order $1$.\(^1\) $C$ can be found exactly by means of the approach developed in Ref. 1. In any case, since $W$ is exponentially small, the approximate breather (3) is, in fact, very stable, and it may be quantized semiclassically. The quantization problem has also been solved in Ref. 3: The quantized values $\epsilon_n$ of the amplitude are

$$\epsilon_n = \gamma n/16 - \frac{1}{3} (\gamma n/16)^3 + O((\gamma n/16)^5),$$

where $\gamma$ is a small coupling constant, and $n$ is a quantum number, $1 \ll n \ll 16/\gamma$. From the semiclassical viewpoint, the emission rate (5) gives the rate $\Gamma$ of the radiative transition $n \rightarrow n - 3$ between the quantized levels: $\Gamma \approx W/3 \gamma$. The small-amplitude breather described by Eq. (1) may be quantized in a similar way. The result is

$$\epsilon_n = 3\gamma/2 - 25(25\gamma)^3/8 + O((\gamma n)^5),$$

In all cases the exponentially small factor in $W$ is $\exp(-\pi k/\epsilon)$, $k$ being the radiation wave number. $k/\epsilon$ is proportional to the ratio of the breather’s size $\sim \epsilon^{-1}$ to the radiation wavelength $\lambda = 2\pi/k$. If a perturbation contains its own length scale $L$, the mentioned ratio changes into $L/\lambda$. In particular, if $L \leq 1$, the energy emission rate is not exponentially small, i.e., a breather is not very long lived. An example of a perturbation of this kind is $a\delta(x)\sin\phi$, for which $L = 0$. We have recently found the corresponding

$$W = 625(3\sqrt{2}/16) a^2 (\epsilon^2 - a^2/4)^3,$$

which is valid for both $a \ll \epsilon$ and $a \approx \epsilon$ (in the latter case the breather’s amplitude is, in fact, not $\epsilon$ but $\epsilon'(= \epsilon - a/2)$. Under the action of this perturbation, the amplitude decays at $t \rightarrow \infty$ according to the law $\epsilon'(t) \sim (a^2 t)^{-1/2}$, while in the case of (1) and (2) the decay is much slower, $\propto (\log t)^{-1}.1$

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