Polarization Oscillations and Soliton Stability in Birefringent Optical Fibres

D. Anderson¹, Yu. S. Kivshar² and M. Lisak³

¹ Institute for Electromagnetic Field Theory, Chalmers University of Technology, S-412 96 Göteborg, Sweden
² Institute for Low Temperature Physics and Engineering, 47 Lenin Avenue, 310164 Kharkov, USSR

Received September 27, 1990; accepted in revised form October 11, 1990

Abstract

Propagation of short pulses in linearly birefringent optical fibres is considered analytically with the help of a variational (Lagrangian) approach for two coupled nonlinear Schrödinger equations. For nearly equal amplitudes of the polarization pulses we study the trapping of solitons due to the nonlinear coupling in the case of strong birefringence, and we find also the amplitude threshold for the trapping when the solitons are produced by a symmetric output pulse. In the case of small birefringence we propose an analytical explanation of the soliton instabilities observed earlier in numerical simulations. The analysis demonstrates instability of the fast polarization mode and stability of the slow one, the effect having a threshold which depends on birefringence. Our results may also be applied to an elliptically birefringent fibre when the birefringence is large.

1. Introduction

It is well-known that solitons in single-mode optical fibres have been theoretically predicted and also observed experimentally (see, e.g., Ref. [1]). Solitons are nonlinear pulses that propagate without broadening in the anomalous dispersion region and may be approximately described by the nonlinear Schrödinger (NLS) equation for the effective one-component field amplitude [1]. However, the propagation equation is independent of the initial polarization state only in a perfectly circular optical fibre where orthogonal polarizations are degenerate. A real fibre is not circular, and the single mode is split into two polarization modes. This splitting may be created by introducing some asymmetry to give an elliptical core or by random perturbations along the fibre length. The nonlinear refractive index leads to an interaction between the two modes of the birefringent fibre which is described by two nonlinearly coupled NLS equations. (See, e.g., Refs. [2] and [3].)

The propagation of solitons in birefringent optical fibres was discussed recently in a number of papers [3–17]. In particular, many authors (see, e.g., Refs. [5, 8, 11]) studied the pulse evolution numerically under the assumption that the two polarizations exhibit identical group velocities and dispersive coefficients. However, interesting new effects arise in the case of different group velocities [4, 6, 17]. Indeed, while linear birefringence leads to a substantial splitting of two polarizations due to a group velocity difference, the Kerr nonlinearity which leads to a strong interaction between the partial polarization pulses may compensate the splitting and give rise to soliton trapping.

The threshold effect of the soliton trapping was studied numerically by Menyk [4, 6] for equal and different amplitudes of the polarization pulses, and in the case of equal amplitudes it was explained analytically in Ref. [17] on the basis of soliton theory. It is important to note that the trapping of the pulses belonging to different nonlinear modes of a multi-mode optical fibre was first predicted by Hasegawa [18] who used an elegant analogy between the model and a quantum mechanics problem, but the results of Ref. [17] explain in more detail the numerically observed dependence of the threshold amplitude of the input pulse on the birefringence parameter. Recently, Islam et al. [14] have demonstrated experimentally the trapping of orthogonally polarized solitons in highly birefringent optical fibres and they have found excellent agreement with Menyk’s numerical simulations.

The pulse splitting in real fibres may be induced by birefringence of randomly varying orientation. For this case, Mollenauer et al. [15] showed numerically that solitons of any pulse width can avoid splitting or excessive broadening for small polarization dispersion parameter. They also observed production of a significant amount of dispersive wave radiation from the soliton state.

The analytical description [17] and numerical results [4, 6, 14] are based on a model of the two-mode fibre in the form of two nonlinearly coupled NLS equations. This model describes a birefringent fibre when rapidly oscillating terms arising from the nonlinear polarization are ignored [4, 6]. The present paper aims to describe analytically the pulse propagation in birefringent fibres on the basis of a more complete model taking into account all terms caused by the nonlinear polarization. In particular, we demonstrate that the Menyk equations for the birefringent fibre are valid in the case when the input pulses have nearly equal phases and the birefringence is not too small. Using the variational approach as elaborated, e.g., in Refs. [19, 21] (this approach is a simple and more convenient version of the soliton perturbation theory [23]) we calculate the interaction energy between solitons of simple polarizations and, after the averaging over the fast oscillations, we obtain a condition for soliton trapping. This condition is similar to that obtained in Ref. [17], but it arises for the averaged soliton parameters. The approach also allows us to consider the case of an elliptically birefringent fibre.

Within the framework of the variational approach, we also study the propagation of the linearly polarized soliton pulses in birefringent fibres when the input pulse consists of a pulse of a simple polarization (an analogue of the one-component NLS equation pulse). We analyse the stability of such pulses and explain instabilities of the polarization modes observed in numerical simulations [5, 11]. The main difference between this case and the previous one is that oscillating nonlinear polarization terms are included. These terms create an instability of the fast polarization mode in the case of small birefringence.
The paper is organized as follows: In Section 2 we introduce the basic equations for birefringent fibres and in Section 3 the variational approach is used to analyse the intermode interaction between the solitons belonging to different polarization modes. In Section 4 we discuss the case in which the partial pulses in each polarization have almost equal amplitudes. Section 5 is devoted to an analysis of the case of strongly different amplitudes of the linearly polarized modes, and their stability. Section 6 concludes the paper.

2. Basic equations

As was demonstrated by Menyuk [3], pulse propagation in a linearly birefringent optical fibre is described by the coupled nonlinear equations,

\[
i \left( \frac{\partial U}{\partial x} + k^* \frac{\partial U}{\partial t} \right) - \frac{1}{2} k^* \frac{\partial^2 U}{\partial t^2} + \frac{\chi}{2} \left( |U|^2 + \frac{3}{2} |V|^2 \right) U + \frac{\chi}{6} V^2 U^* \exp \left[ -2i(k_0 - l_0)x \right] = 0 \quad (1a)
\]

\[
i \left( \frac{\partial V}{\partial x} + \Gamma \frac{\partial V}{\partial t} \right) + \frac{1}{2} \Gamma \frac{\partial^2 V}{\partial t^2} + \frac{\chi}{2} \left( |U|^2 + |V|^2 \right) V + \frac{\chi}{6} U^2 V^* \exp \left[ 2(k_0 - l_0)x \right] = 0 \quad (1b)
\]

where \( k_0 = k(\omega_0) \) and \( l_0 = k(\omega_0) \) are the wave numbers of the two polarization modes evaluated at the carrier frequency \( \omega_0 \). The derivatives \( k^* = \partial k(\omega_0)/\partial \omega_0, k^* = \partial^2 k(\omega_0^2, \Gamma = \partial l(\omega_0), \text{ and } \Gamma = \partial^2 l(\omega_0) \) are also evaluated at \( \omega = \omega_0 \). The parameter \( \chi \) is the Kerr coefficient with a geometric factor due to the finite size of the core taken into account. The functions \( U \) and \( V \) represent the amplitudes of the partial pulse envelopes of each polarization (fast and slow respectively).

To normalize the equations (1) we first assume that \( k^* = \Gamma \) and use variables closely related to standard variables (see, e.g., Refs. [1, 3])

\[
\xi = \frac{\pi x}{2\omega_0}, \quad \zeta_0 = \frac{\pi \omega_0^2 \gamma^2}{D(\lambda_0)}, \quad t_0 = 0.568 \tau, \quad s = \frac{1}{l_0} \left( \frac{1}{2} \frac{\partial \phi}{\partial t} \right), \quad \bar{\chi}_0 = \frac{2}{k^* + \Gamma}, \quad k^* = \Gamma = \frac{\lambda_0}{2\pi \omega_0}, \quad u = \left( \frac{r}{2} \right)^{1/2} U, \quad v = \left( \frac{r}{2} \right)^{1/2} V, \quad \delta = \frac{k^* - \Gamma}{2|k^*|}, \quad R = \frac{8\pi \nu}{\lambda_0}, \quad t_0, \quad (2a)
\]

where \( \tau \) is the FWHM of the pulse intensity. Then eqns. (1) become

\[
i \left( \frac{\partial u}{\partial \xi} + \delta \frac{\partial u}{\partial \xi} \right) + \frac{1}{2} \frac{\partial^2 u}{\partial \xi^2} + (|u|^2 + B|v|^2)u
+ A v^* \exp \left( -i R \delta \xi \right) = 0 \quad (3a)
\]

\[
i \left( \frac{\partial v}{\partial \xi} - \delta \frac{\partial v}{\partial \xi} \right) + \frac{1}{2} \frac{\partial^2 v}{\partial \xi^2} + (B|u|^2 + |v|^2)v
+ A u^* \exp (i R \delta \xi) = 0 \quad (3b)
\]

where

\[
B = \frac{\delta}{2}, \quad A = \frac{1}{2}, \quad A + B = 1
\]

Equations (3) can be transformed into another form by means of the following transformations [5]

\[
u = \frac{1}{2i} e^{i \xi} (\bar{a} + \bar{b}) \quad (5)
\]

\[
u = \frac{1}{2i} e^{i \xi} (\bar{a} - \bar{b}) \quad (5)
\]

where

\[
\kappa = \frac{1}{4} R \delta.
\]

The functions \( \bar{a}(\xi, s) \) and \( \bar{b}(\xi, s) \) satisfy the equations

\[
i \left( \frac{\partial \bar{a}}{\partial \xi} + \delta \frac{\partial \bar{a}}{\partial \xi} \right) + \frac{1}{2} \frac{\partial^2 \bar{a}}{\partial \xi^2} + \kappa \bar{b} + \frac{1}{2} [B|\bar{a}|^2 + (1 + A)|\bar{b}|^2] \bar{a} = 0, \quad (7a)
\]

\[
i \left( \frac{\partial \bar{b}}{\partial \xi} + \delta \frac{\partial \bar{b}}{\partial \xi} \right) + \frac{1}{2} \frac{\partial^2 \bar{b}}{\partial \xi^2} + \kappa \bar{b} + \frac{1}{2} [B|\bar{b}|^2 + (1 + A)|\bar{a}|^2] \bar{b} = 0, \quad (7b)
\]

where we have used the condition \( A + B = 1 \). The functions \( \bar{a} \) and \( \bar{b} \) are related to the real pulse amplitudes as follows:

\[
\bar{a} = u e^{i \xi} + iv e^{-i \xi}, \quad \bar{b} = u e^{i \xi} - iv e^{-i \xi},
\]

and

\[
|u|^2 = \frac{1}{4} [|\bar{a}|^2 + |\bar{b}|^2 + 2 \text{Re}(\bar{a} \bar{b}^*)], \quad (9a)
\]

\[
|v|^2 = \frac{1}{4} [|\bar{a}|^2 + |\bar{b}|^2 - 2 \text{Re}(\bar{a} \bar{b}^*)]. \quad (9b)
\]

Equations (5) to (8) will be used to analyse the stability properties of simple polarizations.

Similar equations arise in the case of the so-called elliptically birefringent fibres [23, 24], where the dynamical equations for the two polarizations may be presented in the above normalized coordinates as follows [25]

\[
i \left( \frac{\partial u}{\partial \xi} + \delta \frac{\partial u}{\partial \xi} \right) + \frac{1}{2} \frac{\partial^2 u}{\partial \xi^2} + (|u|^2 + B|v|^2)u
+ Cu^* \exp \left( -i 4 \xi \right) + D[v^2 u^* \exp (2i \xi)]

+ (2(|u|^2 + |v|^2)v \exp (-2i \xi) = 0, \quad (10a)
\]

\[
i \left( \frac{\partial v}{\partial \xi} - \delta \frac{\partial v}{\partial \xi} \right) + \frac{1}{2} \frac{\partial^2 v}{\partial \xi^2} + (B|u|^2 + |v|^2)v
+ Cu^* \exp (4i \xi) + D[v^2 u^* \exp (-2i \xi)

+ (|u|^2 + 2|v|^2)u \exp (2i \xi) = 0 \quad (10b)
\]

where

\[
B = \frac{2(1 + \epsilon \sin^2 \theta)}{2 + \epsilon \cos^2 \theta}, \quad C = \frac{\epsilon \cos^2 \theta}{2 + \epsilon \cos^2 \theta}, \quad D = \frac{\epsilon \sin \theta \cos \theta}{2 \cos^2 \theta}.
\]

\( \theta \) is the ellipticity angle, and \( \epsilon = \chi(\omega_0, -\omega_0) ; \chi(\omega_0, -\omega_0) \), where \( \chi(\omega_1, \omega_2, \omega_3) \) is the Fourier transform of the nonlinear polarization function (for details see Ref. [25]). A linearly birefringent fibre corresponds to \( \theta = 0 \), while a circularly birefringent fibre corresponds to \( \theta = \pi/2 \). As a result, \( B \) takes the values \( B = B \leq 2 \), when \( \epsilon \approx 1 \).

According to Refs. [3, 6] and references therein, for \( \tau = 5 \text{ps} \) the typical values for \( \delta \) are: \( \delta = 0.3 - 0.3 \), and with \( R \approx 1.4 \times 10^4 \) this yields \( R \delta > 1 \) over the entire range of \( \delta \).
Hence the final temporal terms of eqs. (3) and (10) are rapidly oscillating and, as we will demonstrate below, may be neglected after averaging. For \( \tau = 250 \text{fs} \) typical values are \( \delta = 0.013 - 0.13 \) and \( R = 700 \) and it follows that \( R \delta \leq 1 \) when \( \delta \leq 1.4 \times 10^{-3} \). Thus, when the birefringence is small, the last terms in eqs. (3), (10) can no longer be ignored. These terms lead to an instability in which the faster-moving partial pulse transfers energy to the slower (see, e.g., Ref. [3]). We will consider both of the cases analytically using the variational version of the soliton perturbation theory.

3. Variational approach

To investigate the dynamics of soliton pulses in birefringent fibres we will use the Lagrangian variational approach (see, e.g., Refs. [19–21]). In the variational approach the coupled NLS equations (3) are restated as a variational problem in terms of the Lagrangian,

\[
\delta \left[ \int_{-\infty}^{+\infty} d\xi \int_{-\infty}^{+\infty} ds L \right] = 0,
\]

\[
L = L_u + L_v + L_{\text{mw}} + \Delta L,
\]

where

\[
L_u = \left( \frac{\partial u^*}{\partial \xi} u - \frac{\partial u}{\partial \xi} u^* \right) + i \delta \left( \frac{\partial u^*}{\partial s} u - \frac{\partial u}{\partial s} u^* \right) - \frac{1}{2} \left| \frac{\partial u}{\partial s} \right|^2 + \frac{1}{2} |u|^4,
\]

\[
L_v = \left( \frac{\partial v^*}{\partial \xi} v - \frac{\partial v}{\partial \xi} v^* \right) - i \delta \left( \frac{\partial v^*}{\partial s} v - \frac{\partial v}{\partial s} v^* \right) - \frac{1}{2} \left| \frac{\partial v}{\partial s} \right|^2 + \frac{1}{2} |v|^4,
\]

\[
L_{\text{mw}} = B |u|^2 |v|^2,
\]

\[
\Delta L = \frac{1}{4} A (u^2 v^* e^{i \delta \xi} + u^* v^2 e^{-i \delta \xi}).
\]

In the absence of mutual coupling we will consider the single soliton solutions of different polarizations,

\[
u = 2 \eta_1 \text{ sech} \left[ 2 \eta_1 (s - M_1) \right] \exp \left[ 2 i C_1 (s - M_1) + i D_1 \right], \quad \text{(15a)}
\]

\[
v = 2 \eta_2 \text{ sech} \left[ 2 \eta_2 (s - M_2) \right] \exp \left[ 2 i C_2 (s - M_2) + i D_2 \right], \quad \text{(15b)}
\]

where \( \eta_1 \) and \( \eta_2 \) \((j = 1, 2)\) are arbitrary constants and

\[
M_1 = 2 C_1 + \delta \xi, \quad M_2 = 2 C_2 - \delta \xi, \quad D_1 = 2 (\eta_1^2 + C_1^2) \xi.
\]

Now assuming that the interaction is weak in the sense that the pulses preserve their soliton character and the interaction changes only the soliton parameters, we can formulate the reduced variational problem [cf. eq. (12)],

\[
\delta \left[ \int_{-\infty}^{+\infty} \langle L \rangle d\xi \right] = 0
\]

where

\[
\langle L \rangle = \int_{-\infty}^{+\infty} L ds = \langle L_u \rangle + \langle L_v \rangle + \langle L_{\text{mw}} \rangle + \langle \Delta L \rangle.
\]

Here

\[
\langle L_u \rangle = -4 \eta_1 \left( -2 C_1 \frac{dM_1}{d\xi} + \frac{dD_1}{d\xi} \right) - 8 \delta \eta_1 C_1
\]

\[
+ 8 \eta_1 \left( \frac{\partial u}{\partial s} u - \frac{\partial u^*}{\partial s} u^* \right)
\]

\[
+ \delta \left( \frac{\partial u^*}{\partial s} u - \frac{\partial u}{\partial s} u^* \right) + \delta \left( \frac{\partial u}{\partial s} u^* - \frac{\partial u^*}{\partial s} u \right) - \frac{1}{2} \left| \frac{\partial u}{\partial s} \right|^2 + \frac{1}{2} |u|^4
\]

\[
+ \left( \frac{\partial u^*}{\partial s} u - \frac{\partial u}{\partial s} u^* \right) - \frac{1}{2} \left| \frac{\partial u}{\partial s} \right|^2 + \frac{1}{2} |u|^4
\]

\[
\langle \Delta L \rangle = 16 B \eta_1^2 \eta_2^2 \int_{-\infty}^{+\infty} ds \text{ sech}^2 z_1 \text{ sech}^2 z_2
\]

\[
\langle L_v \rangle = 16 B \eta_1^2 \eta_2^2 \int_{-\infty}^{+\infty} ds \text{ sech}^2 z_1 \text{ sech}^2 z_2
\]

\[
\langle L_{\text{mw}} \rangle = 16 B \eta_1^2 \eta_2^2 \int_{-\infty}^{+\infty} ds \text{ sech}^2 z_1 \text{ sech}^2 z_2 \cos (\Lambda + 4 \kappa \xi),
\]

\[
\langle \Delta \rangle = 16 A \eta_1^2 \eta_2^2 \int_{-\infty}^{+\infty} ds \text{ sech}^2 z_1 \text{ sech}^2 z_2
\]

where \( \Lambda = 2 C_1 (s - M_1) + D_1 - 2 C_2 (s - M_2) - D_2 \).

We can now derive the variational equations with respect to the parameter functions. Using eq. (17), we obtain (j = 1, 2)

\[
\frac{\delta \langle L \rangle}{\delta D_j} = 0 \Rightarrow 4 \frac{d \eta_j}{d \xi} + ( -1)^{j+1} \frac{\delta \langle \Delta \rangle}{\delta \xi} = 0,
\]

\[
\frac{\delta \langle L \rangle}{\delta M_j} = 0 \Rightarrow \frac{d}{d \xi} \left( -8 A C_j \right) + \frac{\delta \langle L \rangle}{\delta \eta_j} + \frac{\delta \langle \Delta \rangle}{\delta \xi} = 0,
\]

\[
\frac{\delta \langle L \rangle}{\delta \eta_j} = 0 \Rightarrow -4 \left( -2 C_j \frac{dM_j}{d\xi} + \frac{dD_j}{d\xi} \right) + ( -1)^{j+1} 8 \delta \eta_j
\]

\[
+ 8 \eta_j^2 (\eta_j - C_j^2) + \frac{\delta \langle L \rangle}{\delta \eta_j} + \frac{\delta \langle \Delta \rangle}{\delta \xi} = 0
\]

Equation (20) yields the energy conservation law

\[
\frac{d}{d \xi} (\eta_1 + \eta_2) = 0 \Rightarrow \eta_1 + \eta_2 = \text{const.}
\]

Equations (20)–(23) form the basis of the variational approach, and they allow us to obtain a set of equations for the solitons’ parameters.

The variational approach may also be applied to the case of the reduced eqs. (7). In this situation, the exact solutions of the system, \( a = b \) or \( a = -b \), correspond to the pulses of the separate polarizations, or [see eqs. (5)]. Changing the notations,

\[
\tilde{a} \rightarrow \frac{\sqrt{B}}{\sqrt{b}} a, \quad \tilde{b} \rightarrow \frac{\sqrt{B}}{\sqrt{b}} b,
\]

we obtain the main part (uncoupled terms) of the equations (7) exactly as in eqs. (3). Then looking for a variational ansatz for the functions \( a \) and \( b \) in the form (25) where indices "1" and "2" correspond to the \( a \)-field and the \( b \)-field, respectively, we may use directly eqs. (20)–(23). The Lagrangian corresponding to the system of equations (7), has the form,

\[
L = \frac{i}{2} \left( \frac{\partial u}{\partial s} a^* - \frac{\partial u^*}{\partial s} a \right) - \frac{1}{2} \left| \frac{\partial u}{\partial s} \right|^2 + \frac{1}{2} |a|^4
\]

\[
+ \frac{i \delta}{2} \left( \frac{\partial b}{\partial s} a^* - \frac{\partial a^*}{\partial s} b \right),
\]

\[
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\]
\[ L_b = \frac{i}{2} \left( \frac{\partial b}{\partial s} b^* - \frac{\partial b^*}{\partial s} b \right) - \frac{1}{2} \frac{\partial b^2}{\partial s^2} + \frac{1}{2} \left| \frac{\partial b}{\partial s} \right|^2 + \frac{i \delta}{2} \left( \frac{\partial a}{\partial s} b^* - \frac{\partial b^*}{\partial s} a \right), \]  
\[ L_{ab} = \frac{(1 + A)}{B} \left| a \right|^2 \left| b \right|^2, \]  
\[ \Delta L = \kappa (b a^* + a b^*), \]  
[eqs. (14)]. We will use the system (7) to analyze the polarization instabilities in the case of small birefringence, e.g., at typical values of \( \delta \lesssim 10^{-3} \). In such a situation, the last terms on the RHS of eqs. (26a, b) may be ignored. Then the Lagrangian (26) at \( \delta = 0, \kappa \neq 0 \) is similar to that considered above, [eqs. (26) and eqs. (14)] if we take into account the change \( B \rightarrow (1 + A)B \) and use \( \Delta L \) as defined in eq. (26d). As a result, the variational eqs. (20)–(23) are valid in the considered case after the substitution

\[ \langle L_{ab} \rangle \rightarrow \langle L_{ab} \rangle = \frac{(1 + A)}{B} \left[ \int_{-\infty}^{+\infty} ds \left| a \right|^2 \left| b \right|^2 \right] \]

\[ = 8(1 + A) \int_{-\infty}^{+\infty} ds \operatorname{sech}^2 z_1 \operatorname{sech}^2 z_2, \]

the parameters \( \Lambda, z_1, \) and \( z_2 \) being defined in eqs. (19d) and (19f).

To study the stability of the separate polarization pulses, we need to consider the solutions (15) for the functions \( b \) and \( a \) with equal and different phases. The case \( \eta_1 = \eta_2, M_1 = M_2, c_1 = c_2, \) and \( D_1 = D_2 \) corresponds to the input pulse \( u \neq 0, v = 0 \). In the opposite case, when \( u = 0, v \neq 0 \), we have to choose \( \eta_1 = \eta_2, M_1 = M_2, c_1 = c_2, \) and \( D_1 - D_2 = \pm \pi \).

Numerical simulations of such a stability are reported in Refs. [5, 11].

4. Almost equal polarization pulses

4.1. Soliton trapping

Let us consider almost equal polarizations, i.e.

\[ z_2 = \frac{\eta_2}{\eta_1} z_1 + 2 \eta_2 (M_1 - M_2) \approx z_1 + \Delta, \]

\[ \Delta = 2 \eta_2 (M_1 - M_2), \]

\[ \eta_1 \approx \eta_2 \approx \eta = \frac{1}{2} (|\eta_1| + |\eta_2|) \gg |\eta_1 - \eta_2|. \]

In this case the functions \( \langle L_{ab} \rangle \) and \( \langle \Delta L \rangle \) may be calculated exactly. Indeed, straightforward calculations yield

\[ i \int_{-\infty}^{+\infty} dx \operatorname{sech}^2 x \operatorname{sech}^2 (x + \Delta) \]

\[ = \frac{2}{\eta} \left( \Delta \coth \Delta - 1 \right) \]

and, as a result,

\[ \langle L_{ab} \rangle = 16 B \eta^4 i \int_{-\infty}^{+\infty} dx \operatorname{sech}^2 x \operatorname{sech}^2 (x + \Delta) \]

\[ = \frac{32 B \eta^4 (\Delta \coth \Delta - 1)}{\sinh^2 \Delta}, \]

\[ \langle \Delta L \rangle = 16 A \eta^4 i \int_{-\infty}^{+\infty} dx \coth \Delta (\Lambda + 4 \kappa \xi), \]

\[ = \frac{32 A \eta^4 (\Lambda \coth \Delta - 1)}{\sinh^2 \Delta} \cos (\Lambda + 4 \kappa \xi). \]

Finally, we obtain the following system of the equations:

\[ \frac{dv}{d\xi} = 16 A \eta^4 j(\Lambda) \sin \chi, \]

\[ \frac{d\mu}{d\xi} = 16 A \eta^4 j(\Lambda) \left( B + A \cos \chi \right), \]

\[ \frac{d\Delta}{d\xi} = 4 \eta \mu + 4 \eta \delta, \]

\[ \frac{d\phi}{d\xi} = 4 (\eta v - C \mu), \]

where

\[ \chi = \frac{c_t}{\eta} \Lambda + \phi \approx 4 \kappa \xi \]

and the following notations are used:

\[ v = \eta_1 - \eta_2, \mu = C_1 - C_2, \]

\[ \phi = D_2 - D_1, \Delta = 2 \eta (\mu_1 - \mu_2), \]

\[ f(x) = \frac{x \coth x - 1}{\sinh^2 x}, \]

\[ f'(x) = \frac{dx}{\sinh x} \]

According to Menyuk [6], for the pulse duration \( \tau \approx 5 \) fs estimates yield \( \delta \approx 0.3-3.0 \) and \( \kappa \approx (1/4) R \delta \approx 10^{-1} - 10^{-4} \gg 1 \) over the entire range of the parameters. As a result, when birefringence is large, the parameter \( \kappa \) in eqs. (34), (35) has rapidly oscillating multipliers. Let us consider the slowly varying dynamics of the soliton parameters in the presence of these fast oscillations. To consider such a situation, when \( dG/d\xi \approx 4 \kappa G \), where \( G = \{ v, \mu, \Delta, \phi \} \) is the set of the relative parameters of the solitons, we will average over the fast oscillations. If we introduce the notations

\[ \bar{G} = \frac{2 \kappa}{\pi} \int_0^{2\pi} d\xi G(\xi), \]

the resulting evolution equations may be presented in the form [eqs. (34)–(38)]

\[ \frac{d\bar{v}}{d\xi} = 0, \]

\[ \frac{d\bar{\phi}}{d\xi} = 4 (\bar{\eta} v - C \bar{\mu}) \]

and

\[ \frac{d\bar{\Delta}}{d\xi} = 16 B \eta^3 j'(\bar{\Lambda}), \]

\[ \frac{d\bar{\phi}}{d\xi} = 4 \eta \bar{\mu} + 4 \eta \delta, \]

where \( \bar{\Delta}, \bar{\mu}, \bar{\nu}, \bar{\phi} \) denote mean values of the parameters. Equations (42), (43) may be considered separately and they are similar to those derived in Ref. [17]. The equations (41)–(43) arise also in the case of an elliptically birefringence fibre described by eqs. (10). When the rapidly oscillating terms \( \sim C, \sim D \) are ignored, the resulting equations in the reduced form (see Ref. [25]) are:

\[ i \left( \frac{\partial u}{\partial \xi} - \delta \frac{\partial u}{\partial s} \right) + \frac{1}{2} \frac{\partial^2 u}{\partial s^2} + (|u|^2 + B|u|^2) v = 0, \]

\[ i \left( \frac{\partial v}{\partial \xi} + \delta \frac{\partial v}{\partial s} \right) + \frac{1}{2} \frac{\partial^2 v}{\partial s^2} + (|v|^2 + B|v|^2) u = 0. \]
Equations (42), (43) and (40) describe the soliton trapping in birefringent fibres. To analyse the problem, we rewrite the system (42), (43) in the more convenient form [cf. Ref. 17],

\[
\frac{d\tilde{\Delta}}{d\tilde{c}} = 4\eta\tilde{\mu} + 4\eta\tilde{\nu},
\]

\[
\frac{d\tilde{\mu}}{d\tilde{c}} = - \frac{1}{4\eta} \frac{\partial}{\partial \Delta} U_{int}(\tilde{\Delta}),
\]

by introducing the interaction energy

\[
U_{int}(\tilde{\Delta}) = 128Bn^4 \left[ \frac{1}{3} \cosh \tilde{\Delta} \sinh^{-1} \frac{\Delta}{\sinh \tilde{\Delta}} \right].
\]

For small \(\tilde{\Delta}\) the function \(U_{int}(\tilde{\Delta})\) may be presented as follows:

\[
U_{int}(\tilde{\Delta}) \approx \frac{128Bn^4}{15} \tilde{\Delta},
\]

and describes small relative oscillations of the soliton positions with a frequency \(\omega_0^2 = (256/15)Bn^4\). These oscillations must be slow as compared to the fast ones with the frequency 4\(\kappa\) [see eqs. (34)–(38)], otherwise the equations (45), (46) for the averaged parameters are not valid. As a result, the condition for the applicability of such an approach is

\[
\frac{16}{15} Bn^4 \ll \kappa^2
\]

It is important that as \(|\tilde{\Delta}| \to \infty\), the interaction energy \(U_{int}(\tilde{\Delta})\) tends to the finite value,

\[
(U_{int})_{max} = \frac{128Bn^4}{3}
\]

From eqs. (45), (46) we can obtain the effective conservation law,

\[
\frac{1}{2} \left( \frac{d\tilde{\Delta}}{d\tilde{c}} \right)^2 + U_{int}(\tilde{\Delta}) = \text{const},
\]

The constant value (the total energy) may be found from the “initial” conditions at \(\tilde{\Delta} = 0\), \(U_{int}(0) = 0\) and \((d\tilde{\Delta}/d\tilde{c})_{\tilde{c}=0} = 4\eta\tilde{\nu}\) [see eq. (45)]. As a result, eq. (50) describes, in the case \(E_{in} < U_{int}\) the trapping of the polarization solitons into a bound pulse. The threshold condition for the trapping may be written as follows,

\[
1/4(4\eta\tilde{\nu})^2 < U_{max} = 128/3 Bn^4
\]

i.e.,

\[
\delta < 4\eta \frac{\sqrt{B}}{\sqrt{3}}.
\]

Thus, when birefringence is large, it leads to soliton trapping due to the mutual interaction between the polarization modes. The pulse trapping has been observed experimentally in highly birefringent linearly polarized optical fibres [14].

4.2. The amplitude threshold

One of the interesting aspects of the problem under consideration is the generation of localized pulses in birefringent fibres by means of the coupling of two different polarization (bound states). In the linear limit, when the nonlinear coupling in eqs. (44) is absent, linear wave pulses produced by the symmetric input pulse

\[
u(0, s) = \nu(0, s) = A_0 \text{ sech } s
\]

will propagate in opposite directions with the velocities (in the s-space) \(\pm \delta\), and the input pulse will split. Indeed, for \(A_0 \ll 1\), when equations (44) may be linearized, the asymptotic solutions of the decoupled equations for the input pulse (52) have the form (cf. Ref. [6]),

\[
u(\zeta, s) = \frac{\sqrt{\pi}}{\sqrt{1-(1-i)\frac{A_0}{2}e^{\pm i \kappa s^2} \text{ sech} \left[ \frac{\pi}{2 \zeta} (s - \delta \zeta) \right]}}
\]

The solutions (53) display a broadening and decoupling of the linear polarization pulses. Nonlinearity produces an effective interaction between the polarizations and will lock together the partial pulses if the initial amplitude is above some threshold (\(A_0)_{thr}\) (see Refs. [4, 6, 17]). This threshold may be obtained analytically within the framework of the approach proposed in Ref. [17], which is based on two assumptions: The first assumption is a relation between \(A_0\) and the amplitudes of the generated (symmetric) solitons, \(\eta_0\). This relation may be obtained exactly in the symmetric case \(\upsilon = \nu\) using results of the inverse scattering transform [26] (see details in Ref. [17]),

\[
\eta = A_0 \sqrt{1 + B} - \frac{1}{2}.
\]

The second assumption takes into account the fact that the exact symmetric solution of eq. (44) describing a bound state of two equal polarizations has the form,

\[
u = \nu = \frac{1}{\sqrt{1+B}} 2\eta \text{ sech} (2\eta s) e^{i\omega t}\]

i.e. it has the additional multiplier \((1 + B)^{-1/2}\). This means that the variational pulses (15) may be improved by this multiplier to apply to the case of \(B \approx 1\).

The above assumptions allow us to present the trapping condition (51) in the form:

\[
A_0 > (A_0)_{thr} = \frac{1}{2\sqrt{1+B}} + \frac{1}{4 \sqrt{B}} \delta,
\]

which is valid for \(B \approx 1\). This compares favourably with results of numerical simulations obtained in Refs. [4, 6] (see Ref. [17]) at \(B = 2/3\), when eqns. (44) correspond to linearly birefringent fibres.

Therefore, when the amplitude \(A_0\) of the input pulse is less than the threshold value \((A_0)_{thr}\), the polarizations interact only weakly and the input pulse (52) will decouple into separate polarizations according to eqs. (53). For \([A_0(0)]_{thr} < A_0 < [A_0(\delta)]_{thr}\) at a fixed value of \(\delta\) the input pulse (52) generates two solitons which propagate in opposite directions but their interaction is not strong enough to result in a bound state. However, for \(A_0 > [A_0(\delta)]_{thr}\) the generated solitons form a bound state due to the nonlinear intermode coupling, and separate polarizations oscillate together trapped by the effective potential (47).

5. Stability of linearly polarized modes

In order to apply the variational equations (20)–(23) to the stability analysis of the polarization modes for small birefringence, it is convenient to use the transformed equations (7). Considering the input pulses for the variational approach in
the form (15) but for the $a$-field and $b$-field, we have to change the terms $\langle L_{\text{ae}} \rangle$ and $\langle \Delta L \rangle$ in the Lagrangian (18) [see eqs. (27)]. In particular, for nearly equal pulse amplitudes $\eta_1 \simeq \eta_2 \simeq \eta$, it is easy to calculate the integral in eq. (27b),

\[ I_2(\Delta) = \int_{-\infty}^{+\infty} \frac{1}{2\eta} \frac{dz_1 sech z_1 sech z_2}{\eta \sinh \Delta}. \]

Thus, the terms of the effective Lagrangian including the mutual interactions of the polarization modes have the form,

\[ \langle L_{\text{oe}} \rangle = 16 \frac{(1 + A)}{B} \eta^4 I_2(\Delta) = \frac{32(1 + A)\eta^4(\Delta \coth \Delta - 1)}{B \sinh^2 \Delta}, \]

\[ \langle \Delta L \rangle = 8\kappa \eta^2 \cos \Delta I_2(\Delta) = \frac{8\kappa \eta \Delta \cos \Delta}{\sinh \Delta}. \]

The functions (58), (59) allow us to present the equations for the relative parameters of the $a$- and $b$-solitons in the form [cf. eqs. (34)–(38)]

\[ \frac{dv}{\xi} = 4\kappa \eta g(\Delta) \sin \left( \frac{C}{\eta} \Delta + \phi \right), \]

\[ \frac{dx}{\xi} = 16 \left( \frac{1 + A}{B} \right) \eta^4 f'(\Delta) + 4\kappa \eta g'(\Delta) \cos \left( \frac{C}{\eta} \Delta + \phi \right), \]

\[ \frac{d\Delta}{\xi} = 4\eta \mu, \]

\[ \frac{d\phi}{\xi} = 4(\eta v - C\mu), \]

where the parameters $v, \mu, \phi, \Delta$, and the function $f(x)$ are defined in eqs. (39), (40), and

\[ g(x) = \frac{x}{\sinh x}, \quad g'(x) = \frac{dg(x)}{dx}. \]

Introducing the new phase variable

\[ \Phi = \frac{C}{\eta} \Delta + \phi \]

we may rewrite the system of eqs. (60)–(63) in the form,

\[ \frac{dv}{\xi} = 4\kappa \eta g(\Delta) \sin \Phi, \]

\[ \frac{d\mu}{\xi} = 16 \left( \frac{1 + A}{B} \right) \eta^4 f'(\Delta) + 4\kappa \eta g'(\Delta) \cos \Phi, \]

\[ \frac{d\Delta}{\xi} = 4\eta \mu, \]

\[ \frac{d\phi}{\xi} = 4\eta v, \]

or as a set of two second-order equations

\[ \frac{d^2 \Delta}{\xi^2} = 64 \left( \frac{1 + A}{B} \right) \eta^4 f'(\Delta) + 16\kappa \eta g'(\Delta) \cos \Phi, \]

\[ \frac{d^2 \Phi}{\xi^2} = 16 \kappa \eta^2 g(\Delta) \sin \Phi. \]

Evidently, eqs. (70), (71) describe the motion of a unit-mass particle in the $(\Delta, \Phi)$ plane in the presence of the effective potential

\[ W(\Delta, \Phi) = -64 \frac{1 + A}{B} \eta^4 f(\eta^2) - 16\kappa \eta^2 g(\Delta) \sin \Phi \]

and, therefore, the system (70), (71) may be presented in the form of the classical equations of motion,

\[ \frac{d^2 \Delta}{d\xi^2} = -\frac{\partial W(\Delta, \Phi)}{\partial \Delta}, \quad \frac{d^2 \Phi}{d\xi^2} = -\frac{\partial W(\Delta, \Phi)}{\partial \Phi}, \]

where $\xi$ corresponds to the time variable. The effective potential (72) and the system (70), (71) at $\kappa \neq 0$ and $(1 + A)/B = 0$ formally coincide with those for the soliton interaction in tunnel-coupled optical fibres (see Ref. [16]).

The equilibrium positions of the potential (72) are

\[ \Delta = 0, \quad \sin \Phi = 0. \]

The trivial solution (73) of eqs. (70), (71) corresponds to a bound state of two $(a$- and $b$-solitons (15) with equal parameters belonging to different polarization modes.

An investigation of small oscillations in the vicinity of eqs. (73) yields the equations

\[ \frac{d^2 \Delta}{d\xi^2} = -\left[ 64 \left( \frac{1 + A}{B} \right) \eta^4 + \frac{16}{3} \kappa \eta^2 (-1)^n \right] \Delta, \]

\[ \frac{d^2 \psi}{d\xi^2} = (-1)^n 16\kappa \eta \psi, \quad \psi = \Phi - \Phi_0, \]

where we have used the expansions of the functions $f(x)$ and $g(x)$ for small $x$,

\[ g(x) \approx 1 - \frac{x^2}{6}, \quad f(x) \approx \frac{1}{3} \left( 1 - \frac{x^2}{10} \right). \]

Here, $\Phi_0$ gives the solutions of the equation $\sin \Phi_0 = 0$, i.e., $\Phi_0 = \pi n, n = 0, 1, 2, \ldots$ which characterise the equilibrium points

\[ \Delta = 0, \quad \Phi = \Phi_0^{(n)} = 2\pi m, \quad m = 0, \pm 1, \ldots, \]

\[ \Delta = 0, \quad \Phi = \Phi_0^{(n)} = \pi + 2\pi m, \quad m = 0, \pm 1, \ldots. \]

According to eqs. (74), (75) the eigenfrequencies corresponding to eqs. (74), (75) are

\[ \omega_\Delta^2 = 64 \frac{1 + A}{15} \eta^4 + \frac{16}{3} (-1)^n \kappa \eta^2, \]

\[ \omega_\Phi^2 = (-1)^n \kappa \eta^2. \]

As a result, an absolutely stable bound state occurs at $\Phi = \Phi_0^{(n)}$, i.e., a bound state of solitons with opposite phases, provided $\omega_\Delta^2 > 0, \omega_\Phi^2 > 0$, i.e., when [see eqs. (77)]

\[ \eta > \eta_c = \frac{5\kappa}{4} \frac{B}{1 + A}. \]

The inequality (78) is the main result of this section. It means that at fixed fibre parameters there is a threshold in the pulse amplitude in order to avoid instability with respect to a mismatch in pulse positions.

Equations (70), (71), (77), (78) are also useful to investigate the stability of the simple $a$- and $b$-polarizations. Numerical simulations of the stability were carried out in Ref. [5] for the cases $u = 0$ and $v \neq 0$ or $u \neq 0$ and $v = 0$, and those results may be compare with our analysis. Using eqs. (5) and
(25) we find the relations between \( u, v \)-fields and \( a, b \)-fields

\[
\begin{align*}
    u &= \frac{1}{\sqrt{2B}} e^{-i\xi} (a + b), \\
    v &= \frac{1}{i\sqrt{2B}} e^{i\xi} (a - b),
\end{align*}
\]  

(79)

which means that the case \( a = \pm b \) corresponds exactly to the input pulses of simple polarizations considered in Ref. [5].

To study their stability in eqs. (15), written in terms of \( a \) and \( b \), we put \( c_i = c_j = 0 \), \( M_i = M_j \) and \( \eta_1 = \eta_2 = \eta \), which corresponds to \( u \)- and \( v \)-pulses in the form of the solitons,

\[
\begin{align*}
    u &= \frac{4n}{\sqrt{2B}} \text{sech} [2\eta (s - M)] e^{-i\xi} e^{i(\theta_1 + \theta_2)/2} \cos (\phi/2) \\
    v &= -\frac{4n}{\sqrt{2B}} \text{sech} [2\eta (s - M)] e^{i\xi} e^{i(\theta_1 + \theta_2)/2} \sin (\phi/2)
\end{align*}
\]  

(80a)

where \( \phi \) is defined in eq. (39). The output pulse (80a) at \( \phi = 0 \) corresponds to the case \( u = u(s, \xi), v = 0 \) (fast mode), and at \( \phi = \pm \pi \) to the case \( u = 0, v = \pm iu(s, \xi) \) (slow mode) where \( u(s, \xi) \) is the soliton solution of the coupled equations related to simple polarizations. According to eqs. (66)-(69) or eq. (70), (71) the dynamics of the phase \( \phi \) introduced in eqs. (80) is described by the equation [cf. eq. (71) at \( \Delta = 0 \)],

\[
\frac{d^2 \phi}{ds^2} = 16\kappa \eta^2 \sin \phi.
\]  

(81)

Equation (81) demonstrates that the fast mode \((|u|^2 = |u|^2, v = 0)\) is unstable and it will evolve into a stable (slow) mode \((u = 0, |v|^2 = |u|^2)\). This is in agreement with numerical simulations performed in Ref. [5]. Since the total energy of the system under consideration is conserved, polarization oscillations of the soliton pulses in the system may be observed. Indeed, when the pulse (80) is launched close to the unstable fast mode \( (\phi = 0) \), the polarization state would move away and undergo large amplitude oscillations around the stable slow mode \( (\phi = \pm \pi) \). This effect is similar to that studied in the case when one neglects all effects of pulse dispersion, and considers only continuous wave (CW) interactions, in which case the evolution equations for the two modes can be solved analytically [27, 28]. However, unlike CW interaction between the modes, dispersion leads to radiation being emitted simultaneously from the oscillating polarization state and the energy switching between the modes will tend to concentrate the pulse energy near the stable (slow) mode, i.e., the pulse oscillation is damped by radiation.

Numerical simulations [5] demonstrated that the fast mode is unstable provided \( \kappa \leq 1.2 \). For larger values of \( \kappa \) the fast mode becomes stable in the sense that the oscillations between the two modes disappear [5]. Our analytical results demonstrate, unlike the results of Ref. [5], the other sense of the instability. Indeed, according to our formula (38), the slow mode is stable provided

\[
\kappa < \kappa_{\text{cr}} = \frac{\frac{1}{2} A}{B}.
\]  

(82)

When \( \kappa > \kappa_{\text{cr}} \) the fast and slow modes are both unstable and the instabilities of both modes may give a new stable polarization state, e.g., the state which was considered in the previous section when the polarization amplitudes are equal.

The result (82) may be presented as a function of the fibre parameter \( \sigma = B/2A \) which is between 0.5 and 3 (see Ref. [5]). Using the relation \( A + B = 1 \), [see eq. (4)] we may find the expressions \( A = 1/(2\sigma + 1) \) and \( B = 2\sigma/(2\sigma + 1) \) which yield

\[
\kappa_{\text{cr}} = \frac{1}{2} \eta^2 \frac{\sigma + 1}{\sigma}.
\]  

(83)

At \( \sigma = 0.5 \) the critical value given by eq. (83) is \( \kappa_{\text{cr}} = 2.4\eta^2 \), and at \( \sigma = 3 \), eq. (83) yields \( \kappa_{\text{cr}} = 16/15\eta^2 \approx 1.1\eta^2 \). The results are in good agreement with the numerical simulations reported in Ref. [5] where the unstable fast mode was observed at \( \sigma = 0.5 \) for \( \kappa \approx 1.2 \) [but corresponds to \( \eta = 1/\sqrt{2} \) in eq. (83)], and the instability occurred at lower values of \( \kappa \) when \( \sigma = 3 \).

6. Concluding remarks

In conclusion, we have considered analytically the interaction between different polarizations in linearly and elliptically birefringent optical fibres. If the input pulse has nearly equal polarizations, \( u \approx v \), so that \( \arg (|u|^2 u) \approx \arg (|v|^2 v) \approx \arg (u) \) or \( \arg (|u|^2 u) \approx \arg (|v|^2 v) \approx \arg (v) \), the exponential terms in eqs. (3) and (20) are, in fact, rapidly oscillating when the birefringence parameter \( R_0 \) [see eq. (2c)] is large. In such a case, the averaged equations for the relative parameters of the interacting solitons correspond to the situation when the oscillating terms may be neglected directly in eqs. (3), (20), and the main effect is the group velocity mismatch of the polarization modes. Differences in group velocities lead to the nonlinear effects of soliton trapping and the existence of an amplitude threshold which were predicted numerically [4] and have been observed experimentally [14]. The variational approach developed in this paper, allows us to explain analytically these effects for linearly and elliptically birefringent fibres when the birefringence is large.

In the case \( u \neq 0 \) and \( v \neq 0 \) (or \( u \neq u \)), the phases of the separate polarizations may be strongly changed due to the birefringence [in fact, due to the last (oscillating) terms in eqs. (3) and (10) which strongly affect phases of the pulses]. The effect is important experimentally for small birefringence parameter \( R_0 \) when the faster-moving partial pulse is unstable and it transfers its energy to the slower polarization mode, the total energy of the pulses being constant. As a result, when the instability occurs, the solitons oscillate between the two polarizations near the slow mode analogously to the effect for the CW interaction. In this case differences in the group velocities of the polarization modes are not so important as in the previous case of strong birefringence. Our analytical results for the latter case are in a good agreement with numerical simulations [5] where the instability of the fast mode was observed provided \( \kappa < \kappa_{\text{cr}} \). This critical value \( \kappa_{\text{cr}} \) is predicted analytically by our variational approach, and implies that for \( \kappa \approx \kappa_{\text{cr}} \) the slow mode is also unstable, and the simple polarizations tend to a new stable state.

Finally, the instability problem may be considered from another point of view. According to our result (78), the critical condition for the birefringence parameter \( \kappa \) may be formulated as a condition for the pulse amplitude, \( \eta \). In such a formulation, for fixed fibre parameters, \( \kappa \) and \( \sigma \) [see eq. (83)], there is the critical amplitude \( n_0^2 = \frac{1}{2} (\kappa \sigma)(1 + \sigma) \) [cf. eq. (78), (83)] corresponding to instability. In fact, for \( \eta^2 < n_0^2 \) the birefringence is large in the sense that, under the condition (48), the pulse dynamics is similar to that considered in section 4. For \( \eta^2 > n_0^2 \) the slow polarization mode

\[
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\]
is stable and soliton oscillations between the two polarizations may be observed.

Acknowledgements

One of the authors (Yu. S. Kivshar) would like to thank the Institute for Electromagnetic Field Theory for hospitality. The authors thank Professor Curtis R. Menyuk for useful discussions related to the paper.

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