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Optics Communications 221 (2003) 199–209

OPTICS COMMUNICATIONS

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# Exact localized and periodic solutions of the discrete complex Ginzburg–Landau equation

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Received 6 March 2003; received in revised form 14 April 2003; accepted 14 April 2003

**Abstract**

We study, analytically, the discrete complex cubic Ginzburg–Landau (dCCGL) equation. We derive the energy balance equation for the dCCGL and consider various limiting cases. We have found a set of exact solutions which includes as particular cases periodic solutions in terms of elliptic Jacobi functions, bright and dark soliton solutions, and constant magnitude solutions with phase shifts. We have also found the range of parameters where each exact solution exists. We discuss the common features of these solutions and solutions of the continuous complex Ginzburg–Landau model and solutions of Hamiltonian discrete systems and also their differences.


*PACS:* 42.65.−k; 42.65.Sf; 42.65.Tg; 46.10.+z; 47.54.+r

*Keywords:* Dissipative discrete solitons; Discrete complex Ginzburg–Landau equation

## 1. Introduction

Discrete solitons in nonlinear lattices have been the focus of considerable attention in diverse branches of science [1]. Discrete solitons are possible in several physical settings, such as biological systems [2], atomic chains [3,4], solid state physics [5], electrical lattices [6] and Bose–Einstein condensates [7]. Recently, the existence of discrete solitons in photonic structures (in arrays of coupled nonlinear optics waveguides [8–13] and in a nonlinear photonic crystal structure [14]) was announced and has attracted considerable attention in the scientific community. Photonic crystals, which are artificial microstructures having photonic

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doi:10.1016/S0030-4018(03)01457-3

bandgaps, can be used to precisely control propagation of optical pulses and beams. They are very useful for optical components such as waveguides, couplers, cavities and optical computers. It is possible to make discrete waveguides using photonic crystals. In this situation, 'discrete solitons' appear naturally and have interesting properties. Many scientists believe that the discrete solitons can have an important role in this technology.

The discrete nonlinear Schrödinger (dNLS) equation is

$$\mathrm{i}\frac{\mathrm{d}\psi_n}{\mathrm{d}t}+\frac{D}{2}(\psi_{n+1}-2\psi_n+\psi_{n-1})+\beta_1|\psi_n|^2\psi_n=0, \tag{1}$$

where $\psi_n$ are complex variables defined for all integer values of the site index $n$. The term $\psi_{n+1}-2\psi_n+\psi_{n-1}$ plainly approximates a second derivative term for a continuous system and so physically represents diffraction. A simple transformation [8] eliminates the term $-2\psi_n$, thus indicating that what is occurring is nearest-neighbour coupling. Hence, a realistic discrete system features diffraction-type effects.

The dNLS equation was used by Christodoulides and Joseph [9] to model the propagation of discrete self-trapped beams in an array of weakly coupled nonlinear optical waveguides. In such an array, when low intensity light is injected into one, it will couple to more and more waveguides as it propagates, thereby broadening its spatial distribution (diffraction). High intensity light changes the refractive index of the input waveguides through the Kerr effect and decouples them from the rest of the array. Certain light distributions propagate while retaining a fixed spatial profile among a limited number of waveguides. These are discrete spatial solitons. Experimental results for optical waveguide arrays, confirming the validity of the model, have been reported by Eisenberg et al. [10].

There are many works relating to the dNLS equation and some its variants [15–21]. It is well known that the standard DNLS equation (1) is not completely integrable. The integrable discrete nonlinear Schrödinger equation (Ablowitz–Ladik (AL) system)

$$\mathrm{i}\frac{\mathrm{d}\psi_n}{\mathrm{d}t}+\frac{D}{2}(\psi_{n+1}-2\psi_n+\psi_{n-1})+|\psi_n|^2(\psi_{n+1}+\psi_{n-1})=0 \tag{2}$$

was found by using the inverse scattering method [22].

The AL system has $N$-soliton solutions and a rich mathematical structure. Although the AL system is 'unphysical', analysis of the AL system gives us useful information to study the non-integrable DNLS.

Most studies related to discrete solitons are directed at conservative systems, i.e., those that preserve energy. However, dissipative systems are more common in nature, so further studies on discrete dissipative systems are certainly required. Ravoux et al. [23] studied the discrete analog of the complex cubic Ginzburg–Landau equation having pattern formation phenomena in mind. In particular, they studied plane wave instability in such systems. Recently, Abdullaev et al. [24] studied the discrete analogue of the complex cubic–quintic Ginzburg–Landau equation with a more general form for the nonlinear terms. Using a perturbation technique, they found a soliton solution which is valid at small values of the dissipative terms for this equation. Efremidis and Christodoulides also studied a different complex cubic–quintic Ginzburg–Landau equation [25]. They found that discrete solitons of the complex cubic–quintic Ginzburg–Landau equation have several features that have no counterparts in either the continuous limit or in other conservative discrete models. We also study a discrete equation, similar to that in [24,25], but exclude the quintic nonlinearity. However, we derive exact solutions which are valid at arbitrary values of dissipative terms for this equation. Hence, we do not restrict ourselves to perturbation values. In particular, we consider a model of a dissipative system, viz. the following discrete complex Ginzburg–Landau equation (dCGLE):

$$\mathrm{i}\frac{\mathrm{d}\psi_n}{\mathrm{d}t}+\left(\frac{D}{2}-\mathrm{i}\beta\right)(\psi_{n+1}-2\psi_n+\psi_{n-1})+(1-\mathrm{i}\epsilon)|\psi_n|^2(\psi_{n+1}+\psi_{n-1})=\mathrm{i}\delta\psi_n. \tag{3}$$

The nonlinear term differs from that in the system studied in [23]. In particular, when $\beta = \epsilon = \delta = 0$, Eq. (3) is reduced to the AL system, rather than to the dNLS.

The nonlinear plane wave solutions of Eq. (3) are defined by

$$\psi_n = A e^{i(kn-\omega t)}. \tag{4}$$

The amplitude $A$, the wave number $k$, and the frequency $\omega$ satisfy the nonlinear dispersion relation

$$|A|^2 = \frac{2\beta(1-\cos(k))-\delta}{2\epsilon\cos(k)} \geqslant 0 \quad \left(k \neq \frac{\pi}{2} + N\pi\right), \tag{5}$$

$$\omega = \left(D - \frac{2\beta}{\epsilon}\right)(1-\cos(k)) + \frac{\delta}{\epsilon}. \tag{6}$$

Nevertheless, the continuous limit of (3) is the complex Ginzburg–Landau equation (CGLE)

$$i\frac{d\psi}{dt} + \left(\frac{D}{2} - i\beta\right)\psi_{xx} + (1-i\epsilon)|\psi_n|^2\psi_n = i\delta\psi, \tag{7}$$

which has many applications in describing non-equilibrium systems, phase transitions, and wave propagation phenomena. When $\epsilon = \beta = \delta = 0$, Eq. (7) is reduced to the NLS equation. Eq. (7) has exact solutions [26,27] and many interesting properties [28–30].

The Hirota method can be applied to obtain selected exact solutions of Eq. (3). In order to do this we substitute

$$\psi_n(t) = \phi_n e^{-i\omega t} = \frac{g_n}{f_n} e^{-i\omega t},$$

$$\psi_n^*(t) = \phi_n^* e^{i\omega t} = \frac{g_n^*}{f_n} e^{i\omega t},$$

with $f_n$ real, into Eq. (3). As a result, we obtain the multi-linear form

$$\left(\omega - i\delta - 2\left(\frac{D}{2} - i\beta\right)\right) f_{n+1} f_n f_{n-1} g_n + \left(\frac{D}{2} - i\beta\right) f_n^2 f_{n-1} g_{n+1}$$
$$+ \left(\frac{D}{2} - i\beta\right) f_{n+1} f_n^2 g_{n-1} + (1-i\epsilon) f_{n-1} g_{n+1} g_n g_n^* + (1-i\epsilon) f_{n+1} g_n g_n^* g_{n-1} = 0.$$

Then, the standard procedure of the Hirota method can be used to obtain the exact solutions listed in the following sections. Before presenting the exact solutions, we note that all solutions of (3) satisfy some general relations that can be derived directly from the equations. One of them is the energy balance equation.

## 2. Energy balance equation

The system that we are studying is dissipative. Hence, we do not expect that any quantity composed of $\psi_n$'s will be conserved. However, as in the case of continuous CGLE [30], there may be one or more balance equations. To find one, we define a 'modified energy', $Q$:

$$Q = \sum_n \log\left(\frac{D}{2} + |\psi_n|^2\right).$$

We expect to have $Q$ conserved for the Ablowitz-Ladik case Eq. (2). It is not exactly an extension from the continuous case. However, if we set $D = 2$ and we have $|\psi_n|^2 \ll 1$, then $Q = \sum_n \log(1 + |\psi_n|^2) \approx \sum_n |\psi_n|^2$, so it is then directly proportional to the total energy of the system. An interesting physical implementation is an electrical lattice where each node consists of a capacitor and an inductor [15,16]. Then $\psi_n$ is the node voltage $V_n$, but the charge on the $n$th (nonlinear) capacitor is $Q_n = \log(1 + |V_n|^2)$ and so the AL system conserves total charge, $Q = \sum_n Q_n$, but not $\sum_n |V_n|^2$.

In [32], we derived two balance equations for the continuous CGLE, including the one for the energy. Here we follow a similar procedure to write down the rate of change of $Q$. After relatively simple calculations, we find

$$\frac{1}{2}\frac{\mathrm{d}Q}{\mathrm{d}t} = (\delta - 2\beta)\sum_n \frac{|\psi_n|^2}{\frac{D}{2} + |\psi_n|^2} + \sum_n \frac{\beta + \epsilon|\psi_n|^2}{\frac{D}{2} + |\psi_n|^2}\,\mathrm{Re}[\psi_n^*(\psi_{n+1} + \psi_{n-1})]. \tag{8}$$

The terms on the right-hand side of this equation allow us to identify the contributions of the terms in the discrete equation to the overall change in energy. If $\epsilon = \beta = \delta = 0$, then $Q$ is conserved, as expected. For stationary solutions which are the point of our interest in this work, the sum of all contributions on the right-hand side must be zero, although each term can be non-zero. This condition can serve as an additional tool for finding stationary solutions.

## 3. Discrete solitons and quasi-periodic solutions with constant phase

For the solutions in this section $\psi_n$ has constant phase along the chain. As a result, we apply the constraint on the equation parameters

$$D = \frac{2\beta}{\epsilon}. \tag{9}$$

Then the frequency is given by

$$\omega = \frac{\delta}{\epsilon}. \tag{10}$$

These two relations are for zero chirp, so they no longer hold if $\phi_n$ is complex.

*Bright soliton.* Using the Hirota method, we can find the explicit solution for the fundamental soliton with a constant phase across its profile

$$\phi_n = \frac{1}{2}\sqrt{\frac{\delta}{\epsilon}\left(\frac{\delta}{\beta} - 4\right)}\,\mathrm{sech}\left[n\,\mathrm{arccosh}\left(1 - \frac{\delta}{2\beta}\right) + n_a\right]. \tag{11}$$

The argument of the sech function can also be written in terms of the log function. The solution (11) requires $\delta/\beta < 0$ and that $\delta$ and $\epsilon$ have opposite signs. An example is $\delta = 2$, $\beta = -1$, $n_a = 0$, and $\epsilon = -3$ which gives the soliton $\phi_n = \mathrm{sech}[n\,\mathrm{arccosh}(2)]$. The amplitude and the width of the soliton are fixed and defined by the parameters of the equation, as occurs in other dissipative systems. The physical reason for this is the double balance required for the dissipative solitons [30]. However, the constant $n_a$ is a free parameter which influences the shape of the soliton.

Here $n_a$ is an arbitrary real constant, indicating translational invariance along the lattice. Although it seems simple, the translational invariance is not as trivial as in the case of the continuous equation. When $n_a$ is zero, the center of the soliton coincides with a lattice site. In this instance, the solution is symmetric. When $n_a$ is not zero, the soliton center is located between the lattice sites. Then the soliton shape is asymmetric. In this sense, the parameter $n_a$ produces a continuous family of solitons with variable shape. This translational invariance is known to appear for integrable models like the AL system. However, if the model is not integrable, but still Hamiltonian, then this translational invariance is broken [31]. Our results show that the translational invariance is restored in the case of this dissipative system.

*Oscillatory 'sec' solution.* When we consider a different range of the equation parameters, we obtain a solution which has a similar mathematical form, but quite a different pattern. It gives a quasi-periodic oscillation – it does not actually repeat, unless the period of the solution becomes commensurate with the period of the chain. This solution requires $0 < \delta/\beta < 4$ and $\delta/\epsilon < 0$. The solution is

$$\phi_n = \frac{1}{2}\sqrt{\frac{\delta}{\epsilon}\left(\frac{\delta}{\beta} - 4\right)} \sec\left[n \arccos\left(1 - \frac{\delta}{2\beta}\right) + n_a\right]. \tag{12}$$

An example is $\delta = -3$, $\beta = -2$ and $\epsilon = 5$ which gives the solution

$$\phi_n = \frac{1}{2}\sqrt{\frac{3}{2}} \sec\left[n \arccos\left(\frac{1}{4}\right)\right].$$

In the continuous model a solution in terms of sec function would have singularities. In the discrete model the singularity may occur as well. However, this happens only when the solution becomes infinite at the site of the chain. This can be avoided by properly choosing the equation parameters and translation parameter $n_a$.

*Dark soliton.* The dark soliton solution is

$$\phi_n = \sqrt{\frac{-\delta}{2\epsilon}} \tanh\left[n \operatorname{arctanh}\left(\sqrt{\frac{\delta}{2\beta}}\right) + n_b\right]. \tag{13}$$

Here $n_b$ is again an arbitrary constant, showing translational invariance. The parameters $\delta$ and $\epsilon$ must have opposite signs for this solution to exist. On the other hand, $\delta$ and $\beta$ must have the same sign and we actually require $0 < \delta/\beta < 2$.

In the limit of large $n$, the solution reduces to a constant

$$\phi_n(t) = \sqrt{\frac{-\delta}{2\epsilon}}. \tag{14}$$

This can be considered as another independent ('plane wave') solution of dCGLE. The solutions above are analogous to the soliton solutions of the continuous model.

*Oscillatory 'tan' solution.* The above dark soliton solution also has an extended form involving trigonometric functions instead of the hyperbolic ones. Here, the parameters $\delta$ and $\epsilon$ must have the same sign for this solution to exist. On the other hand, $\delta$ and $\beta$ must have opposite signs. The solution is

$$\phi_n = \sqrt{\frac{\delta}{2\epsilon}} \tan\left[n \arctan\left(\sqrt{\frac{-\delta}{2\beta}}\right) + n_b\right]. \tag{15}$$

An example is $\delta = -3$, $\beta = 1$, and $\epsilon = -2$ which gives the soliton

$$\phi_n = \frac{\sqrt{3}}{2} \tan\left[n \arctan\left(\sqrt{\frac{3}{2}}\right)\right].$$

*Alternating phase bright soliton.* A type of soliton solution which does not have a continuous analog has each site being completely out-of-phase with each of its neighbours. In other words, we have an 'alternating phase' bright soliton. The allowed range of parameters differs from the bright soliton, where each site has a positive value. The solution is

$$\phi_n = \frac{(-1)^n}{2}\sqrt{\frac{\delta}{\epsilon}\left(\frac{\delta}{\beta}-4\right)}\mathrm{sech}\left[n\,\mathrm{arccosh}\left(\frac{\delta}{2\beta}-1\right)+n_c\right], \tag{16}$$

with $n_c$ being an arbitrary translation. This solution requires $\delta/\beta > 4$. As a consequence, $\epsilon$ should have the same sign as $\delta$ (usually negative, as $\delta$ generally represents loss). An example is $\delta = -5$, $\beta = -1$, $n_c = 0$ and $\epsilon = -\frac{5}{4}$, which gives the soliton $\phi_n = (-1)^n\,\mathrm{sech}[n\,\mathrm{arccosh}(3/2)]$.

If the parameter range differs, so that $1-(\delta/2\beta) > 1$ (i.e., $\delta/\beta < 0$), then we have

$$\phi_n = \frac{(-1)^n}{2}\sqrt{\frac{\delta}{\epsilon}\left(\frac{\delta}{\beta}-4\right)}(-1)^n\,\mathrm{sech}\left[n\,\mathrm{arccosh}\left(1-\frac{\delta}{2\beta}\right)+n_c\right],$$

which agrees with the regular bright soliton solution found earlier, Eq. (11), assuming $n_c = n_a$.

*Alternating phase 'sec' oscillatory solution.* The allowed range of parameters differs from the solution equation (16) and this is not a pulse solution. It is oscillatory but not periodic in general. The solution is

$$\phi_n = \frac{(-1)^n}{2}\sqrt{\frac{\delta}{\epsilon}\left(\frac{\delta}{\beta}-4\right)}\ \sec\left[n\ \arccos\left(\frac{\delta}{2\beta}-1\right)+n_c\right]. \tag{17}$$

This solution requires $0 < \delta/\beta < 4$. As a consequence, $\epsilon$ should have the opposite sign to $\delta$.

*Alternating phase dark soliton.*

$$\phi_n = \frac{(-1)^n}{2}\sqrt{\frac{2(\delta-4\beta)}{\epsilon}}\tanh\left[n\,\mathrm{arctanh}\left(\sqrt{2-\frac{\delta}{2\beta}}\right)+n_d\right], \tag{18}$$

with $n_d$ being arbitrary. This solution requires $2 < \delta/\beta < 4$ and that $\epsilon$ and $\delta - 4\beta$ have the same sign.

An example is $\delta = -3$, $\beta = -1$, $n_d = 0$, and $\epsilon = \frac{1}{2}$, which gives the soliton $\phi_n = (-1)^n\tanh[n\,\mathrm{arctanh}(1/\sqrt{2})]$. In the limit of large $n$ this solution is transformed into

$$\phi_n(t) = \frac{(-1)^n}{2}\sqrt{\frac{2}{\epsilon}(\delta-4\beta)}. \tag{19}$$

This solution can be considered as an independent periodic solution of the dCGLE. The phase shift between neighboring sites in this case is $\pi$. The case of a solution with a more general phase shift is given below.

This solution takes the form of the plain dark soliton solution, Eq. (13), if we substitute $(-1)^n\phi_n \to \phi_n$ and $\delta \to 4\beta - \delta$, assuming $n_d = n_b$.

*Alternating phase 'tan' oscillatory solution.* Mathematically, this looks similar to Eq. (18) above but it is not a soliton and is an irregular oscillation. For particular parameters, it can be periodic. It is

$$\phi_n = (-1)^n\sqrt{\frac{(4\beta-\delta)}{2\epsilon}}\tan\left[n\ \arctan\left(\sqrt{\frac{\delta}{2\beta}-2}\right)\right]. \tag{20}$$

This solution requires $\delta/\beta > 4$ and that $\epsilon$ and $4\beta - \delta$ have the same sign.

The above solutions cover much of the possible ranges of the parameters.

*Periodic 'tan' oscillatory solutions.* There are other solutions involving the '*tan*' function which are periodic. One is

$$\phi_n = \sqrt{\frac{\delta-3\beta}{3\epsilon}}\tan\left(\frac{n\pi}{3}\right). \tag{21}$$

This solution requires $(\delta-3\beta)/\epsilon > 0$ . An example is $\delta = 4, \beta = \epsilon = 1$. This gives $\phi_n = (0,1,-1,0,1,-1,0,1,-1,\ldots)$ and the period is 3.

Another related solution is

$$\phi_n = (-1)^n \sqrt{\frac{\beta-\delta}{3\epsilon}} \tan\left(\frac{n\pi}{3}\right). \tag{22}$$

This solution requires $(\beta-\delta)/\epsilon > 0$ . An example is $\beta = 2, \delta = \epsilon = 1$. Then $\phi_n = (0, -1, -1, 0, 1, 1, 0, -1, -1, \ldots)$ and the period is 6.

## 4. Phase-shifting constant magnitude solutions

The dCGLE is invariant relative to a phase shift, i.e., any solution can be multiplied by a complex constant of unit magnitude, viz. $\exp(i\theta)$ for arbitrary $\theta$. We assume this factor throughout, without including it explicitly.

We now consider solutions of the form

$$\psi(m,n;t) = \phi(m,n;t)\exp(-i\omega t) = \sqrt{c(m)}\exp\left(i\pi\frac{n}{m}\right)\exp(-i\omega t), \tag{23}$$

where $m \neq 0$ and $c(m)$ can be taken as real, after accounting for the $\exp(i\theta)$, as noted above. The phase shift between successive sites is $180/m$ degrees. For example, for $m = 2$,

$$\psi(2,n;t) = \sqrt{c(2)}(i)^n \exp(-i\omega t),$$

where $c(2)$ is arbitrary but we require $\delta = 2\beta$ and find $\omega = D$ in the solution.

This is the quadrature (90° advance) case.

For arbitrary $m$ the solution is given by Eq. (23) with

$$c(m) = \frac{1}{2\epsilon}\left[(2\beta-\delta)\sec\left(\frac{\pi}{m}\right) - 2\beta\right],$$

while the frequency is

$$\omega = \omega(m) = D + \frac{1}{\epsilon}(\delta - 2\beta) + 2\left(\frac{\beta}{\epsilon} - \frac{D}{2}\right)\cos\left(\frac{\pi}{m}\right).$$

This conveniently allows us to simplify various solutions. Apart from the $m = 2$ case, there are no constraints on the equation parameters. For $m = 1$, we have

$$\phi(1,n;t) = \sqrt{c(1)}(-1)^n$$

with

$$c(1) = \frac{\delta - 4\beta}{2\epsilon},$$

$$\omega = \omega(1) = 2D + \frac{1}{\epsilon}(\delta - 4\beta).$$

This agrees with the periodic solution given earlier.

The $m = 2$ has to be done separately, since $\cos(\pi/2) = 0$ (see above). For $m = 3$, we have $\phi(3,n;t) = \sqrt{c(3)}(e^{i\pi/3})^n$, where $c(3) = (\beta-\delta)/\epsilon$ and $\omega = \omega(3) = (D/2) + (1/\epsilon)(\delta-\beta)$. Here the phase advance is 60° per site.

Finally, we look at $m = 4$ , where the phase advance is 45°. $\phi(4,n;t) = \sqrt{c(4)}(e^{i\pi/4})^n$. Here

$$c(4) = \frac{(2-\sqrt{2})\beta - \delta}{\sqrt{2}\epsilon},$$

$$\omega = \omega(4) = (2-\sqrt{2})\frac{D}{2} + \frac{1}{\epsilon}((2-\sqrt{2})\beta + \delta).$$

## 5. Elliptic function solutions

Potts [33] has found Jacobi elliptic function solutions of the discrete Duffing equation. We can use a similar procedure to derive elliptic function solutions of the dCGLE. Once again, we take $\phi_n$ real, so Eqs. (10) and (9) apply to the solutions below.

*Jacobi cn function solution* has the form

$$\psi_n(t) = A\,\mathrm{cn}[2nK/p, m]\exp(-\mathrm{i}\omega t), \tag{24}$$

where $K$ is the complete elliptic integral of the first kind ($K(m) = \int_0^{\pi/2}(\mathrm{d}\varphi/\sqrt{1-m\sin^2\varphi})$ where $m$ is modulus $(0 < m < 1)$), and $p$ is an integer ($> 2$).

For this solution $\phi_n = A\,\mathrm{cn}((2/p)nK(m), m)$, we can write down $A^2$ in terms of $m$:

$$A^2 = \frac{2\beta - \delta}{2\epsilon\,\mathrm{cn}[\frac{2}{p}K(m), m]} - \frac{\beta}{\epsilon}, \tag{25}$$

but we can only expand $\mathrm{cn}[(2/p)K(m), m]$ as a simple function of $m$ when $p = 4$. For other values, we need to solve a transcendental equation numerially to get $m$.

As an example we take $p = 3$, and set $\beta = \epsilon = -1, \delta = 3$. We find that the modulus is $m \approx 0.958928$ and the (period 6) solution is

$$\phi_n(n = 0, 1, \ldots) = (3.108, 0.729, -0.729, -3.108, -0.729, 0.729, 3.108, 0.729, \ldots)$$

*Simplified $p = 4$ cn solution.* We can simplify the solution

$$\phi_n = a\,\mathrm{cn}\left[\frac{2}{p}nK(m), m\right], \tag{26}$$

when $p = 4$. Hence, $\phi_n = a\,\mathrm{cn}(\frac{1}{2}nK(m), m)$. This is possible because the required value of $m$ can be written as $m = 1 - \sinh^4 b$, where

$$b = \frac{1}{2}\mathrm{arcsinh}\left(\frac{4\beta}{2\beta - \delta}\right). \tag{27}$$

Then we find the amplitude from

$$a^2 = \frac{1}{2\epsilon}[(2\beta - \delta)\coth(b) - 2\beta]. \tag{28}$$

This produces a sequence of period 8, with the $\phi_n, (n = 0, 1, \ldots)$ being given by

$$a, a\tanh(b), 0, -a\tanh(b), -a, -a\tanh(b), 0, a\tanh(b), a, \ldots$$

with $b$ given above in Eq. (27).

*Jacobi dn function solution* has the form

$$\psi_n(t) = A\,\mathrm{dn}[nK/p, m]\exp(-\mathrm{i}\omega t), \tag{29}$$

where $K$ is the complete elliptic integral of the first kind , $m$ is the modulus $(0 < m < 1)$), and $p$ is an integer ($> 2$). The amplitude $A$ can be written in terms of $m$:

$$A^2 = \frac{2\beta - \delta}{2\epsilon\,\mathrm{dn}\left[\frac{2}{p}K(m), m\right]} - \frac{\beta}{\epsilon}, \tag{30}$$

however, we again need to solve an equation numerically to find $m$.

*Simplification of $p = 4$ dn function solution.* This solution also has an interesting simplified form, since we can use the fact that $\mathrm{dn}(K(m)/2, m) = (1-m)^{1/4}$. For convenience, we let $h = -\delta/\beta$. Then we introduce a hyperbolic function by setting

$$\sinh(b) = \frac{1}{4}[2 + h - \sqrt{(6+h)(h-2)}].$$

Then the solution $\phi_n = a\,\mathrm{dn}((n/2)K(m), m)$, where $m = 1 - \sinh^4 b$, has amplitude $a$ found from

$$a^2 = \frac{2\beta - \delta - 2\beta\sinh(b)}{2\epsilon\sinh(b)}.$$

We need $h = -\delta/\beta > 2$ for these solutions. For large $h$, we note that $m \to 1$.

This period 4 solution, $\phi_n$ $(n = 0, 1, \ldots)$, is given by

$$a, a\sinh(b), a\sinh^2(b), a\sinh(b), a, a\sinh(b), a\sinh^2(b), a\sinh(b), a, a\sinh(b), \ldots$$

with $b$ given above. For example, if $h = 3$, then $\sinh(b) = \frac{1}{2}$, and $m = 15/16$.

The solution is then

$$\phi_n, (n = 0, 1, \ldots) = a, \frac{a}{2}, \frac{a}{4}, \frac{a}{2}, a, \frac{a}{2}, \frac{a}{4}, \frac{a}{2}, a, \frac{a}{2}, \frac{a}{4}, \frac{a}{2}, \ldots$$

*Jacobi sn function solution* is given by

$$\psi_n(t) = A\,\mathrm{sn}[nK/p, m]\exp(-\mathrm{i}\omega t), \tag{31}$$

where $K$ is again the complete elliptic integral of the first kind, and $p$ is an integer ($> 2$). Here the $n = 1$ term provides the transcendental equation.

*Simplified $p = 4$ sn solution.* For the elliptic function solution

$$\phi_n = a\,\mathrm{sn}\left(\frac{2}{p}nK(m), m\right),$$

as with the corresponding one with cn (see Eq. (26) above), we find that $\phi_1$ can only be written as a simple function of $m$ when $p = 4$. In general the period is $2p$, but for general $p$, we need to find $m$ numerically.

Here, with $p = 4$, we start by defining

$$g = \left(1 - \frac{\delta}{2\beta}\right)^2,$$

and $y = \frac{1}{2}[g + \sqrt{g^2 + 4g}]$. Then $b = \mathrm{arcsinh}(\sqrt{y})$ and

$$a^2 = \frac{\cosh(b)}{\epsilon}[2\beta - \delta - \beta\cosh(b)],$$

and $m = 1 - y^2$.

This again produces a sequence of period 8, with the $\phi_n$ $(n = 0, 1, \ldots)$ being given by

$$0, a\,\mathrm{sech}(b), a, a\,\mathrm{sech}(b), 0, -a\mathrm{sech}(b), -a, -a\,\mathrm{sech}(b), 0, \ldots$$

with $b$ given above .

*Limit $m \to 1$ of $p = 4$ sn solution.* Using the sn solution, we can obtain $m = 1$ by taking $y = 0$ and hence $g = 0$. This requires $\delta = 2\beta$, and we find that the amplitude is $a = \sqrt{-\beta/\epsilon}$. Taking the limit of the functions, we find the interesting solution

$$\phi_n(n = 0, 1, \ldots) = (0, a, a, a, 0, -a, -a, -a, 0, a, a, a, 0, -a, -a, \ldots).$$

The period is still 8. There is no corresponding cn solution, since then we would need the amplitude to be zero to get $m = 1$.

## 6. Conclusions

We have analysed the dCGLE, and we have found several exact localised and periodic solutions. We have also found the range of parameters where the exact solutions exist. We have found that the width and amplitude of soliton solutions are fixed, due to the double balance in the dissipative system. We can contrast this with the fact that the width and amplitude of a soliton in a Hamiltonian (including integrable) system are variable and that solitons comprise one or two parameter families. On the other hand, it happens that solitons admit translational invariance in the same way as in the integrable Ablowitz–Ladik lattice. As the lattice itself lacks translational invariance, the solitons have an additional degree of freedom in their profile. This result also distinguishes solitons in discrete dissipative lattices from those in Hamiltonian ones, where only fixed positions of the solitons relative to the lattices are allowed.

Another interesting point is that the lattice admits 'sec'-type solutions without the singularities characteristic of continuous models, where the amplitude of such solutions can go to infinity at certain points. In the case of lattice, the points of singularity can be 'between sites' and hence the singularities can be avoided. Discrete lattices also admit solutions with alternating phases which do not exist in continuous models.

Finding exact solutions is an important step in studying discrete lattices, but we should maintain some overall perspective in estimating their significance. In this aspect, we can point out several important issues related to the exact solutions. Firstly, our exact soliton solutions in Section 3 are chirpless. Thus they differ from the exact solutions obtained for the continuous CGLE, where the solutions do have a chirp. We cannot claim, though, that our solutions exhaust the list of possible solutions. Further work is needed to find other possible types of solutions.

Another issue is the stability of the solutions. It is very likely that the exact solutions of the *cubic* discrete CGLE are unstable, as was the case with the continuous *cubic* CGLE. Therefore, quintic terms would have to be added to the equation to make the solitons stable. However, the quintic equation may present more difficulties in terms of deriving exact solutions. This direction of research is also desirable and important. However, the above issues do not diminish the significance of the fact that exact solutions do exist. They may allow further progress in the analysis of discrete dissipative nonlinear systems.

## Acknowledgements

K.M. was supported by a JSPS Fellowship for Young Scientists. N.A. acknowledges support from the Australian Research Council.

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