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Motion and stability properties of solitons in discrete dissipative structures

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Abstract

The discrete complex Ginzburg–Landau (dCGL) equation describes solitons in multiple waveguide structures. We study, numerically, its soliton solutions. We compare stability, translational invariance and motion properties for various cases, including the Ablowitz–Ladik chain, the cubic and two forms of quintic dCGLE.

Discrete models describe important devices in nonlinear optics [1–3]. Among them, the simplest case is the dual-core fiber or nonlinear directional coupler [4]. The number of cores in the fiber, or the number of waveguides in a planar structure, can be more than two, and in this case these structures can function as multi-core switching devices with complicated switching characteristics [4]. The analysis of such devices is based on the theory of discrete nonlinear systems. An early example of such a system is the Ablowitz–Ladik (AL) chain. It is described by the equation

$$i\frac{d\psi_n}{dt} + (\psi_{n+1} - 2\psi_n + \psi_{n-1}) + |\psi_n|^2(\psi_{n+1} + \psi_{n-1}) = 0, \tag{1}$$

which is known to be integrable [5]. Hence, the analysis can be based on the exact results developed in [5]. The latter is known as the inverse scattering method.

When applied to an array of coupled optical waveguides, the nonlinear terms in the discrete nonlinear Schrödinger equations are "local" if only the nonlinearity of the material within the waveguides is taken into account [1]. This means that each equation only contains nonlinear terms related to one particular waveguide. This assumption has generally been

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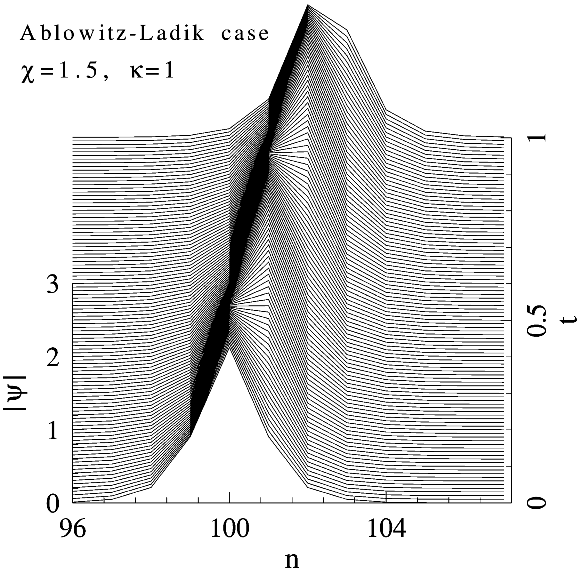

Fig. 1. Moving soliton of the Ablowitz–Ladik equation.

used in the majority of publications [2,3,6]. However, coupling between the waveguides can occur through a nonlinear medium located between the waveguides, or through the substrate. In this instance, we have a "nonlocal" nonlinearity of the type given by the equations (1). This type of nonlinearity models a discrete electrical lattice which was made with nonlinear capacitors [7], and it is not difficult to realize a similar configuration with an optical waveguide array.

Eq. (1) has a two-parameter exact soliton solution, namely:

$$\psi_n(t) = \frac{\sinh(\chi)\exp[i\kappa(n - x_0) + i\omega t]}{\cosh[\chi(n - vt - x_0)]}, \tag{2}$$

where χ and κ are free parameters of the solution, $\omega = 2(1 - \cosh(\chi)\cos(\kappa))$, and the velocity is given by

$$v = \frac{2\sinh\chi\,\sin\kappa}{\chi}.$$

Some remarkable properties of this solution are as follows: (1) the solution has translational invariance, since x_0 is an arbitrary real number, although the lattice itself does not have this invariance; (2) the solution has an arbitrary amplitude, defined by the parameter χ; (3) the solution can move with arbitrary velocity v. A numerical example of such a moving solution is shown in Fig. 1. The shape of the solution changes periodically during propagation, due to the discreteness of the system. The solutions of integrable models are usually neutrally stable.

All the above properties are consequences of the integrability. If we modify the equation, even slightly, by removing its integrability but retaining the Hamiltonian nature of the system, then the translational invariance is lost. The latter is clear because the system itself does not possess translational invariance. Instead, the soliton solution can be located either at a lattice site or between two of them, depending on the value of the potential related to the lateral shift of the solution. As a result, moving solutions also disappear after the modification of the equation if the soliton does not have enough energy to overcome the above potential.

Optical systems are not always Hamiltonian. They can have gain and loss, and in that case are said to be dissipative. A dissipative generalization of the Ablowitz–Ladik system is described by the following equation:

$$\begin{aligned} &i\frac{d\psi_n}{dt} + \left(\frac{D}{2} - i\beta\right)(\psi_{n+1} - 2\psi_n + \psi_{n-1}) \\ &\quad + (1 - i\epsilon)|\psi_n|^2(\psi_{n+1} + \psi_{n-1}) = i\delta\psi_n, \end{aligned} \tag{3}$$

which was considered recently by Abdullaev [8] et al. as a perturbation of the AL system. In particular, they considered the dissipative terms responsible for gain and loss to be small, and studied deformations of the solution (2) under such perturbations. As their analysis showed, soliton solutions do exist. However, the solution obtained in this way is only valid for small values of the parameters responsible for gain and loss.

Exact soliton solutions exist even when the dissipative terms are not small [9]. One example of such a solution is given by $\psi_n = \phi_n e^{-i\omega t}$, where ϕ_n is real. This means that $D = 2\beta/\epsilon$ and $\omega = \delta/\epsilon$, and the solution can be written in the form:

$$\begin{aligned} \phi_n = {}& \frac{1}{2}\sqrt{\frac{\delta(\delta - 4\beta)}{\beta\epsilon}} \\ &\times \operatorname{sech}\left[n \operatorname{arccosh}\left(1 - \frac{\delta}{2\beta}\right) + n_a\right]. \end{aligned} \tag{4}$$

Here n_a is an arbitrary constant, indicating translational invariance. This solution requires $\delta/\beta < 0$. An example is $\delta = 2$, $\beta = -1$, $n_a = 0$ and $\epsilon = -3$ which

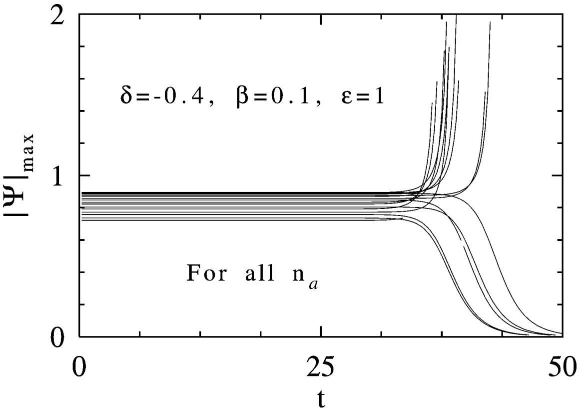

Fig. 2. Evolution of maximum amplitude shows unstable propagation of the soliton solution of the cubic equation.

gives the soliton $\phi_n = \operatorname{sech}[n \operatorname{arccosh}(2)]$. Other soliton solutions with explicit forms are also known [9].

Although it appears simple, the translational invariance is not as trivial as in the case of the corresponding continuous equation. When n_a is zero, the center of the soliton coincides with a lattice site. In this instance, the solution is symmetric. When n_a is not zero, the soliton center is located between two lattice sites. Then the soliton shape can be asymmetric. In this sense, the parameter n_a produces a continuous family of solitons with variable shape. This translational invariance is known to appear for the Ablowitz–Ladik system. However, if the model remains Hamiltonian, but is not integrable, then this translational invariance is lost [11]. However, it is restored in the case of this dissipative system. This fact is not surprising, since dissipative systems do not have any conserved quantities, and correspondingly, there is no potential that changes with soliton position relative to the lattice. Instead, stationary solutions are the result of a double balance—between the diffraction and nonlinearity and between gain and loss. As we can see, these balances can be reached for any position of the soliton relative to the lattice.

An important issue related to discrete dissipative solitons is stability. We have performed numerical simulations on the propagation of the soliton (4) and these show that it is indeed a stationary solution for any value of n_a. However, the solution is not stable, and after a certain distance of propagation, perturbations grow and the soliton either collapses or disappears. Fig. 2 gives an example of a simulation. The solid lines show the evolution of the amplitude at the site of the lattice where the field is maximum. This remains constant over a long distance, but deviates exponentially from the constant when the perturbation effect accumulates. We have looked for stable soliton solutions of Eq. (3) using various initial conditions and for a variety of equation parameters, but so far we have not found any. The actual reason for this result could be that the cubic equation does not allow stable soliton solutions. Either the soliton itself is stable but the background is unstable (when $\delta > 0$) or the background is stable (when $\delta < 0$) but the soliton is unstable. In each case the total solution is unstable. The situation here may be similar to that with the continuous cubic equation, which is known to lack stable solitons, except in very special cases [10] when the soliton and background are neutrally stable (when $\delta = 0$).

Stability can be achieved if we add quintic terms to the system. To see this, let us consider the following equation:

$$\begin{aligned} i\frac{d\psi_n}{dt} &+ \left(\frac{D}{2} - i\beta\right)(\psi_{n+1} - 2\psi_n + \psi_{n-1}) \\ &+ (1 - i\epsilon)|\psi_n|^2(\psi_{n+1} + \psi_{n-1}) \\ &+ (\nu - i\mu)|\psi_n|^4(\psi_{n+1} + \psi_{n-1}) = i\delta\psi_n, \end{aligned} \tag{5}$$

which differs from (3) by the term $(\nu - i\mu)|\psi_n|^4 \times (\psi_{n+1} + \psi_{n-1})$. No analytic solutions for this equation are known. However, we have been able to find stable discrete soliton solutions numerically. Fig. 3 shows two examples of such solutions for the same equation parameters. The solution can be centered on a site or between two sites. In this particular case, the translational invariance of the solution is absent. Furthermore, for a large variety of equation parameters, we were not able to find any asymmetric solution with its center not being either a site or the midpoint between two sites. We note, however, that the translational invariance re-appears for other forms of the quintic dCGLE. For example, we can write the quintic equation in the form

$$\begin{aligned} i\frac{d\psi_n}{dt} &+ \left(\frac{D}{2} - i\beta\right)(\psi_{n+1} - 2\psi_n + \psi_{n-1}) \\ &+ 2(1 - i\epsilon)|\psi_n|^2\psi_n + 2(\nu - i\mu)|\psi_n|^4\psi_n = i\delta\psi_n, \end{aligned} \tag{6}$$

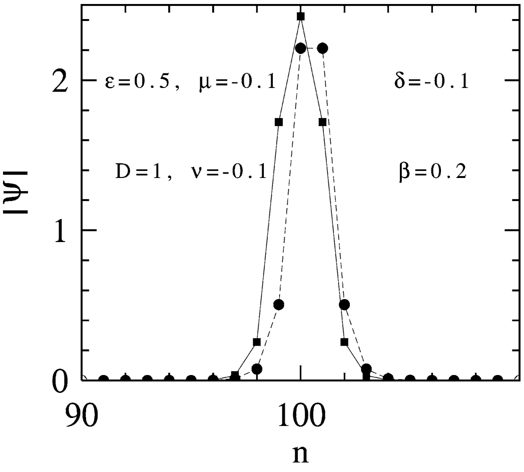

Fig. 3. Examples of soliton solutions of the quintic equation centered at a site and between sites. The equation parameters are the same in both cases and are written in the figure.

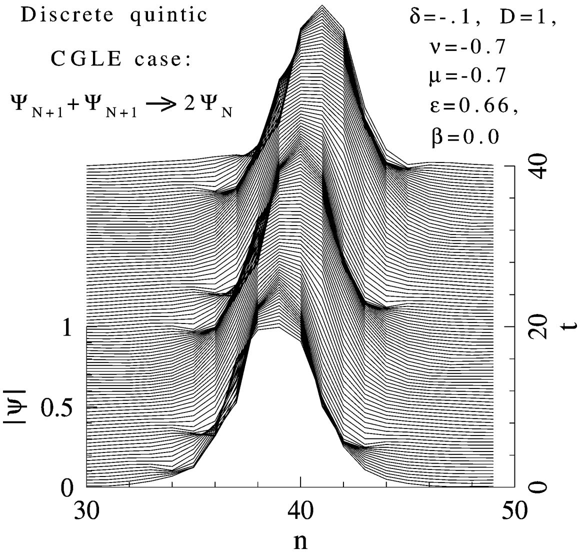

Fig. 4. Moving discrete soliton of the quintic discrete CGLE.

which differs from (5) in the form of its cubic and quintic terms.

In addition to zero-velocity solitons, this equation has moving solitons similar to those of the Ablowitz–Ladik equation. An example is shown in Fig. 4. The motion of the soliton creates periodic changes in its profile. We note that, in a Hamiltonian system, there would be radiation waves emitted from any periodic solution. In our example, radiation waves dissipate due to δ being negative. As a result, the solution is perfectly periodic. Thus, the latter form of the quintic discrete CGLE has soliton solutions with translational invariance. This example shows that, in principle, soliton solutions may have translational invariance in dissipative discrete systems, in contrast to solitons in Hamiltonian systems.

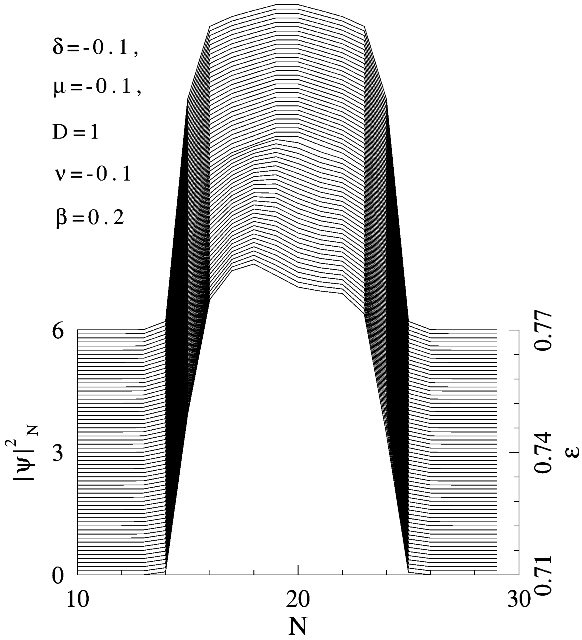

Fig. 5. Soliton profile versus ϵ. The shape changes from being asymmetric to completely symmetric in this range of ϵ. Parameters of the calculation are shown in the figure.

Another result that we obtained with the quintic equation (5) is that asymmetric solitons exist. The existence of asymmetric solutions, in itself, is not unusual. In the case of the continuous CGLE model, these solutions do exist, and the asymmetry results in motion of the solitons with a certain velocity [4]. The discrete lattice, however, can retain asymmetric motionless solitons. Examples of asymmetric soliton profiles are shown in Fig. 5. The shape depends strongly on the equation parameters. By changing ϵ, we can change the shape of the soliton from symmetric to asymmetric. There is a threshold value of $\epsilon = 0.74$, where the stable soliton becomes asymmetric. In each case, the soliton velocity is zero.

In conclusion, we have studied the stability of soliton solutions of the cubic and quintic discrete

Ginzburg–Landau equations. We have compared various cases, including the integrable Ablowitz–Ladik chain, cubic dCGLE and two forms of quintic dCGLE. It turns out that the quintic terms are essential for the stability of these solutions. One of the important conclusions of our study is that solitons in dissipative discrete systems can admit a translational invariance which is similar to that which occurs for the integrable Ablowitz–Ladik chain. Asymmetric, as well as symmetric, solitons can appear in these systems.

Acknowledgements

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